The Least Squares Boundary Residual Method in Electrostatic and Eddy Current Problems

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Abstract—The applicability of the least squares boundary residual method (LSBRM) in electrostatic and eddy current problems has been demonstrated. In each of the treated examples, the method appeared easy to use and highly accurate. This was also the case in eddy currents caused by nonharmonic excitation.

I. Introduction

GENERALLY, all electromagnetic problems can be divided into two classes: deterministic and eigenvalue. The numerical methods for solving both can be grouped in different ways. The most common way is based on the form of problem solution, where the numerical methods can be grouped into three classes. In the first, the solution is given in the form of a set of sought quantities, found in a discreet set of nodal points. The second gives the solution as a set of expressions, each of them valid in a certain subregion. The third gives the solution in the form of a finite series, valid in the whole region in which the solution is required.

The finite difference method and all its improved varations belong to the first class. The finite element method belongs to the second class, since the solution domain is divided, most frequently, in triangle subregions. The solution is found as a set of polinomes, each of them valid in one of the subregions.

The variational approach to the finite difference method is one of the possible variants, while, in the finite element method, this approach is necessary. Moreover, with such an approach it can be shown that these two numerical methods are identical in some cases, such as in one-dimensional and two-dimensional problems with rectangular boundaries.

Although the finite element method is widely accepted, still one can face a lot of difficulties in programming. This method may, in some cases, such as in open boundary problems, give rather inaccurate results.

The common basis of the third class of numerical methods is that the solution is sought in a form of a linear combination of functions. The unknown coefficients are found from the requirement that the residuum of the error function is minimal. Among these methods, which give a continual solution, are the Galerkin method, the Galer-

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kin-Ritz method, the method of moments, and the collocation method. The difference between them is that the developing coefficients and weighting functions are determined and chosen in different ways.

The least squares boundary residual method (LSBRM) is among the methods of the third class. It has been successfully used in the electromagnetic scattering and eigenvalue problems [1]-[3], as well as acoustic wave propagation along a periodic metal grating [4]. A new variant of the method was proposed to analyze planar transmission lines [5]. More recently, by combining the LSBRM and the fast Fourier transform (FFT) algorithms an efficient numerical procedure was demonstrated on diffraction and eigenvalue problems [6]. However, the application of the LSBRM in electrostatic and eddy current problems seems to be less popular than in the microwave domain, although in many cases it may offer many advantages.

II. DESCRIPTION OF THE METHOD

In the interior of the spatial domain, bounded by surface $S = S_1 + S_2$, we shall seek the solution which satisfies the homogenous differential equation

$$\mathcal{L}\big\{\Phi(\vec{r})\big\} = 0. \tag{1}$$

The boundary condition on the part S_1 is of the Dirichlet type, so that

$$\Phi(\vec{r}_1) = \Phi_1(\vec{r}_1). \tag{2}$$

The solution satisfies the Neumman boundary condition on S_2

$$\frac{\partial \Phi(\vec{r}_2)}{\partial n} = \Phi_2(\vec{r}_2) \tag{3}$$

where \vec{r}_1 and \vec{r}_2 are the radius vectors of the points on S_1 and S_2 .

Let $\varphi_k(\vec{r})$ $(k = 1, 2, 3, \cdots)$ be a set of basis functions of the differential equation (1), so the approximate solution may be introduced as

$$\Phi(\vec{r}) = \sum_{k=1}^{N} \alpha_k \varphi_k(\vec{r}). \tag{4}$$

We shall determine the unknown coefficients α_k from the requirement that the integral of quadratic deviation over the boundary surface should be minimal. The quadratic deviation of the approximate solution (4) from

boundary conditions (2) and (3) is

$$\left[\Phi_1(\vec{r}_1) - \Phi(\vec{r}_1)\right]^2$$

and

$$\left[\Phi_2(\vec{r}_2) - \frac{\partial \Phi(\vec{r}_2)}{\partial n}\right]^2. \tag{5}$$

Multiplying the above expressions by the elements of S_1 and S_2 and summing over both surfaces, we get

$$e^{2} = \int_{s_{1}} \left[\Phi_{1}(\vec{r}_{1}) - \Phi(\vec{r}_{1}) \right]^{2} dS + g \int_{s_{2}} \left[\Phi_{2}(\vec{r}_{2}) - \frac{\partial \Phi(\vec{r}_{2})}{\partial n} \right]^{2} dS$$
 (6)

where g is a weighting factor of the Neumman boundary condition.

Inserting the approximate solution (4) into (6), after squaring and interchanging the summation and integration, we find

$$e^{2} = \int_{s_{1}} \Phi_{1}(\vec{r}_{1}) dS + g \int_{s_{2}} \Phi_{2}(\vec{r}_{2}) dS$$
$$-2 \sum_{k=1}^{N} \alpha_{k} b_{k} + \sum_{k=1}^{N} \sum_{l=1}^{N} \alpha_{k} \alpha_{l} a_{kl}$$
(7)

where

$$a_{kl} = \int_{s_1} \varphi_k(\vec{r}_1) \varphi_l(\vec{r}_1) dS + g \int_{s_2} \frac{\partial \varphi_k(\vec{r}_2)}{\partial n} \frac{\partial \varphi_l(\vec{r}_2)}{\partial n} dS$$

(8a

$$b_{k} = \int_{s_{1}} \Phi_{1}(\vec{r}_{1}) \varphi_{k}(\vec{r}_{1}) dS + g \int_{s_{2}} \Phi_{2}(\vec{r}_{2}) \frac{\partial \varphi_{k}(\vec{r}_{2})}{\partial n} dS.$$
(8b)

In order to minimize the integral of quadratic deviation it is necessary to have

$$\frac{\partial e^2}{\partial \alpha_k} = 0. (9)$$

The above requirement leads to the following set of linear equations:

$$\sum_{l=1}^{N} \alpha_{l} \mathbf{a}_{kl} = b_{k} \qquad (k = 1, 2, \dots, N)$$
 (10)

which we shall rewrite in the form

In some problems it may be necessary to express the solution in a more general form

$$\Phi(r) = \sum_{k=1}^{N} \left[\alpha_k \varphi_k(\vec{r}) + \beta_k \psi_k(\vec{r}) \right]$$
 (12)

where $\psi_k(\vec{r})$ is another set of basis functions.

Following the same procedure as above, with the additional requirement that

$$\frac{\partial e^2}{\partial \beta_k} = 0 \tag{13}$$

gives the set of equations for unknown coefficients α_k and β_k

$$\left\| \|a_{kl}\| \cdots \|c_{kl}\| \right\| \left\| \|\alpha_{k}\| \right\| = \left\| \|b_{k}\| \right\|.$$
 (14)

Here, the new coefficients a'_{kl} and b'_k have the same form as a_{kl} and b_k in (8a) and (8b), but with ψ_k instead of φ_k , while c_{kl} are the coupling coefficients between the two sets of basis functions

$$c_{kl} = \int_{s_1} \varphi_k(\vec{r}_1) \psi_l(\vec{r}_1) dS + g \int_{s_2} \frac{\partial \varphi_k(\vec{r}_2)}{\partial n} \frac{\partial \psi_l(\vec{r}_2)}{\partial n} dS.$$
(8a)

Thus we have come to the approximate solution which satisfies the differential equation exactly, while the boundary condition is satisfied in such a way that the integral of total quadratic deviation is minimal.

The first step in the application of this method is the choice of appropriate basis functions, although this choice is, in many cases, quite evident from the geometry of the problem. With well-chosen basis functions, the only practical difficulty appears to be the numerical integration in (8a) and (8b), which is more or less a routine procedure. In the problems where the boundary surface or their parts are plane surfaces, the integrals (8a) and (8b) can usually be found in closed form. Moreover, in some practical problems, at least one part of the boundary is parallel to the coordinate planes, so that the basis functions can be chosen in a form to satisfy the boundary condition on this part of the boundary exactly. In principle, this improves the accuracy of the approximating solution, or at least makes finding the coefficients a_{kl} and b_k easier. The advantage is also that the determinant of (11) is symmetric in relation to the main diagonal, i.e., $a_{kl} = a_{lk}$.

It should be noted that in the most general case, all of the coefficients in (11) are nonzero, since, even if φ_k is a set of orthogonal functions on an arbitrary complex surface S, they become nonorthogonal. In special cases when φ_k and their normal derivatives are orthogonal on the boundary S, all coefficients $a_{kl} = 0$ and $a_{kk} \neq 0$. Thus the LSBRM method can be understood as the expansion of the solution $\Phi(\vec{r})$ on nonorthogonal functions in general, while the expansion on orthogonal functions (e.g., the Fourier series) appears as a special case.

It is quite clear that in two-dimensional problems the domain boundary is a closed line, and consequently the surface integrals in (8a) and (8b) should be replaced by line integrals over the cross-section line.

III. APPLICATION IN ELECTROSTATICS

For Laplace's differential equation in a Cartesian coordinate system, the basis functions have the form

$$\varphi_k(x, y, z) = e^{(k_1x + k_2y + k_3y)}$$
 (16)

where k_1 , k_2 , and k_3 are any complex numbers satisfying condition

$$k_1^2 + k_2^2 + k_3^2 = 0. (17)$$

As an example we shall analyze the two-dimensional electrostatic problem of a two-conductor line placed in a grounded metalic rectangular box (Fig. 1) of infinite extent

The dimensions of the problem are chosen so that the proximity effect is very much present. In the accepted coordinate system, instead of the general basis functions (16), we can take the basis functions of the form

$$V(x, y) = \Phi(x, y) = \sum_{k=1}^{N} \alpha_k \varphi_k$$
$$= \sum_{k=1}^{N} \alpha_k \sin(k\pi x/d) \sinh(k\pi y/d) \quad (18)$$

by which the boundary conditions are satisfied on three walls of the box (x = 0, x = d, and y = 0). The problem can be solved in such a way that half of the real domain is to be considered, so that along the line L_1 (the boundary line of the conductor) the Dirichlet boundary condition should be fulfilled and along L_2 (the rest of boundary line) the Neumann boundary condition is to be met. The coefficients (8a) and (8b) now get the form

$$a_{kl} = \int_{L_1} \varphi_k(x, f(x)) \varphi_l(x, f(x)) dl$$

$$+ g \int_{L_2} \frac{\partial \varphi_k(x, h/2)}{\partial y} \frac{\partial \varphi_l(x, h/2)}{\partial y} dx \quad (19a)$$

$$b_k = \int_{I_*} V\varphi_k(x, f(x)) dl.$$
 (19b)

Here, dl is the element of the line L_1 . The weighting factor g is taken to be g=1. As will be shown later, for time-dependent problems, the role of this factor will be very important.

For a very accurate solution, with dimensions d = h = 2 and a = r = 0.25, only N = 5 modes were needed.

As can be seen from Fig. 2(a), the fulfillment of the boundary condition along L_1 and L_2 is practically perfect, although, bearing in mind the proximity effect and the small number of modes, one would not expect so. The equipotential lines are plotted in Fig. 2(b).

To illustrate the applicability of the method in cylindrical electrostatic problems we shall consider the potential distribution in the capacitor with long cylindric and ring electrodes (Fig. 3). A good, but rather complicated solution of a similar problem is achieved in [7] by using the charge simulation method introduced in [8].

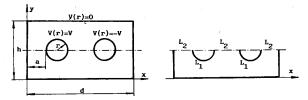


Fig. 1. Two-conductor line in a rectangular box.

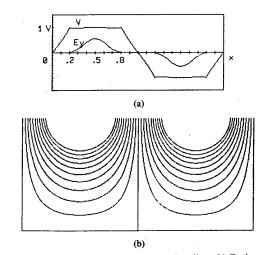


Fig. 2. (a) Potential and field along the boundary line. (b) Equipotential lines.

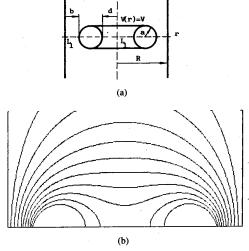


Fig. 3. (a) Cross section of a capacitor with long cylindric and ring electrodes. (b) Equipotential lines.

A solution will be sought in the form

$$V(x, y) = \Phi(r, z) = \sum_{k=1}^{N} \alpha_k \varphi_k$$
$$= \sum_{k=1}^{N} \alpha_k J_0(x_k r/R) e^{-x_k z/R}$$
(20)

where x_k are the roots of Bessel function $J_0(r)$. So, the boundary condition on the cylindric electrode is satisfied exactly. Besides, the requirement $V \to 0$ for $z \to \infty$ is also fulfilled, but the evenness of the solution in respect to z is not satisfied. Thus the coefficients α_k are determined from the requirement that Dirichlet's condition is satisfied along the line L_1 and Neumann's condition along L_2 . Here, we took N=9 modes and for R=2.4, d=1.1, and a=0.4. The equipotential lines were plotted on Fig. 3(b).

The fulfillment of the boundary condition in this case is also very high.

IV. EDDY CURRENT PROBLEMS

The skin effect is a frequent problem in electrical engineering, so a large variety of approaches to its solution has been developed. The skin effect in a multiconductor system was analyzed in [9] by the use of finite elements. The Fredholm integral equation and electric vector potential for eddy currents of a slowly varying field were used in [10]. Another paper [11] also employed the boundary integral equation methods for some electromagnetic field problems. Boundary element techniques for two-dimensional eddy current problems were applied in [12].

The distribution of eddy currents on a cross section of a long linear conductor, which is exposed to the axial harmonic magnetic field of angular frequency ω , is governed by the Helmholtz equation

$$\Delta_{xy} \boldsymbol{H} + k^2 \boldsymbol{H} = 0 \tag{21}$$

with $k = \sqrt{-j\omega\mu\sigma}$. The basis functions of this equation are

$$\varphi_k(x, y) = e^{j(k_1x + k_2y)}, \quad k_1^2 + k_2^2 = k^2.$$
 (22)

The LSBRM method will be applied to the problem of a rather complex cross section determined by the line

$$(x^2 + y^2)^2 - 2c^2(x^2 - y^2) = a^4 - c^4$$
 (23)

where we took a = 1 and c = 1.02.

The boundary condition is $H(L) = H_0$, where H_0 is the amplitude of the external field and L is the boundary line of the conductor. Having in mind the symmetry of the cross section on x and y, instead of the general form (22), basis functions are to be taken as cos functions, so the solution is sought in the form

$$H(x, y) = \Phi(x, y) = \sum_{k=1}^{N} \alpha_k \varphi_k$$
$$= \sum_{k=1}^{N} \alpha_k \cos(k\pi x/A) \cos(\sqrt{j\omega\mu\sigma - (k\pi/A)^2}y)$$

where A is the periodicity of the solution which was taken to be A = 1.6.

From (8a) and (8b) the complex coefficients a_{kl} and b_k are determined. With $\omega\mu\sigma=25$ we took N=13 modes of series (24) and found that the deviation of the field along the boundary is less than 0.5 percent.

Fig. 4(a) shows the lines of constant magnetic field am-

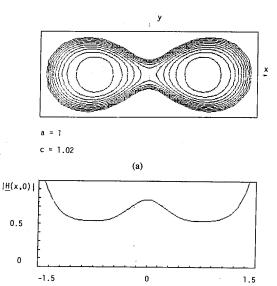


Fig. 4. (a) Lines of constant field amplitudes. (b) Relative field distribution along y = 0.

plitudes, while in Fig. 4(b), we have plotted the relative field distribution along the line y = 0.

V. THE TIME-DEPENDENT PROBLEMS

For eddy currents caused by a nonharmonic magnetic field, we are to solve the more general equation

$$\Delta_{xy}H - \mu\sigma \frac{\partial H}{\partial t} = 0. \tag{25}$$

The basis functions of (25) are

$$\varphi_{mn}(x, y, t) = e^{j(k_m x + k_n y)} e^{-[k^2 t/(\mu \sigma)]}, \qquad k_m^2 + k_n^2 = k^2$$
(26)

where k_m and k_n are arbitrary real numbers. The approximate solution should be sought as

$$H(x, y, t) = \sum_{m=1}^{M} \sum_{n=1}^{M} \alpha_{mn} \varphi_{mn}(x, y, t).$$
 (27)

If we denote the boundary condition (external field on the boundary line of the conductor cross section) by $H_L(x, y, t)$ and the initial condition (field distribution on the cross section in t = 0) by $H_0(x, y)$, the integral of quadratic deviation now becomes

$$e^{2} = \int_{L} \int_{T} \left[H_{L}(x, y, t) - \sum_{m=1}^{M} \sum_{n=1}^{M} \alpha_{mn} \varphi_{mn}(x, y, t) \right]^{2} dl dt$$

$$+ g \int_{s} \left[H_{0}(x, y, 0) - \sum_{m=1}^{M} \sum_{n=1}^{M} \alpha_{mn} \varphi_{mn}(x, y, 0) \right]^{2} dS.$$
 (28)

In the same manner as for system (10) we get a system for the unknown coefficients

$$\begin{vmatrix} a_{11,11} & \cdots & a_{11,MM} \\ \vdots & & \vdots \\ a_{MM,11} & \cdots & a_{MM,MM} \end{vmatrix} \begin{vmatrix} \alpha_{11} \\ \vdots \\ \alpha_{MM} \end{vmatrix} = \begin{vmatrix} b_{11} \\ \vdots \\ b_{MM} \end{vmatrix}$$
 (29)

where

$$a_{mn,kl} = \int_{L} \int_{T} \varphi_{mn}(x, y, t) \varphi_{kl}(x, y, t) dl dt$$

$$+ g \int_{s} \varphi_{mn}(x, y, 0) \varphi_{kl}(x, y, 0) dS$$

$$b_{kl} = \int_{L} \int_{T} H_{L}(x, y, t) \varphi_{kl}(x, y, t) dl dt$$

$$+ g \int_{s} H_{0}(x, y, 0) \varphi_{kl}(x, y, 0) dS. \quad (30)$$

To illustrate the procedure described we shall consider the establishment of a magnetic field in the interior of a linear rhombic cross-section conductor (Fig. 5), the dimensions of which are $a=1,\,b=0.5$, and the electromagnetic constant $\sigma\mu=2.5$.

The conductor is exposed to an axial nonharmonic field

$$H_L(x, y, t) = H_s(1 - e^{-ct})$$
 (31)

with c = 0.5.

The basis function can be chosen, because of symmetry, with respect to the x and y axes, as

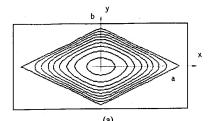
$$\varphi_{mn}(x, y, t) = \cos(m\pi x/2a)\cos(n\pi y/2b)$$

$$\cdot \exp\left[-\frac{(m\pi/2a)^2 + (n\pi/2b)^2}{\sigma\mu}t\right].$$
(32)

By finding the coefficients (30) and solving system (29) with M=7 we get the solution, the accuracy of which with respect to the boundary condition, is no less than 2.2 percent. The best results were achieved with the weighting factor g=0.05. The influence of g on the errors in the boundary and initial condition can be seen from Table I. The smaller weighting factor produces a smaller error in the boundary condition and a greater error in the initial condition and vice versa.

The constant field amplitude lines are plotted in Fig. 5(a) at moment t = 0.1. Fig. 5(b) shows the time dependence of the field in the conductor central point.

It should be noted that for M = 7, because of the double series (27), the number of unknowns in the system (29) is 49. This may give an impression that the method is too extensive for these kinds of problems. However, one should keep in mind that the solving of the problem by some other numerical method such as the finite difference method, requires the discretization of the time interval and finding the nodal field values at each step of time, which is a much more time-consuming procedure.



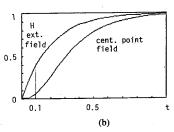


Fig. 5. The establishment of the field in a rhombic conductor. (a) Constant field lines at t = 0.1. (b) Field establishing in the central point.

TABLE I THE INFLUENCE OF g ON THE ERROR

ġ	Error on the Boundary (%)	Error of the Init. Condition (%)
0.005	0.8	5.9
0.05	2.2	1.9
0.5	7.4	0.6

The basis functions (22) and 26) have a simple form and in many cases enable us to find the coefficients in (10) or (29) in closed form. Besides, they have a general character and can be applied to the conductor of any cross section.

VI. CONCLUSION

The applicability of the LSBRM has been demonstrated on several problems in electrostatics (the Laplace equation) and eddy currents, both harmonic and nonharmonic (the Helmholtz and general parabolic equation). In each of them, the method showed a great advantage over the other methods. It gave a continual solution, the programming was easy, and the accuracy was very high, even with a relatively small number of modes. The only difficulty which might have appeared was the choice of appropriate basis functions. However, in many problems this choice is quite obvious. As expected in the nonharmonic eddy current problems, where both space and time boundary conditions were included, the weighting factor played an important role.

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