A Multitime Definition of the Wigner Higher Order Distribution: L-Wigner Distribution

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***Abstract*— A dual form of the Wigner higher order spectra is introduced. Its analysis in the case of multicomponent signals is performed. An efficient distribution for time-frequency signal analysis (L-Wigner distribution) is derived from that analysis. The theory is illustrated on a numerical example.**

I. Introduction

Higher order spectral analysis has been intensively studied during last few years. Higher order statistics known as cumulants and their Fourier transforms (FT), which are known as higher order spectra, are often considered, but we refer here only to the review paper [1] and the references therein. Recently, higher order time-varying spectra have been defined and analyzed [2]. The basic representation in the time-varying higher order spectral analysis is the Wigner higher order spectra. Its definition, of order k, of a complex deterministic signal $x(t)$ is [2]

$$W_k(t,\omega_1,\omega_2,\ldots,\omega_k) = \int_{\tau_1}\int_{\tau_2}\cdots\int_{\tau_k} x^*(t-\alpha) \times \prod_{i=1}^{k}[x(t-\alpha+\tau_i)e^{-j\omega_i\tau_i}d\tau_i]$$
$$\text{with } \alpha = \frac{1}{k+1}\sum_{i=1}^{k}\tau_i. \quad (1)$$

The Wigner higher order spectra, expressed in terms of the FT $X(\omega)$ of signal $x(t)$, is [2]

$$W_k(t,\omega_1,\omega_2,\ldots,\omega_k) = \frac{1}{2\pi}\int_\theta X^*\left(\sum_{i=1}^{k}\omega_i + \frac{\theta}{k+1}\right) \times \prod_{i=1}^{k} X\left(\omega_i - \frac{\theta}{k+1}\right)e^{-j\theta t}d\theta. \quad (2)$$

We will introduce and analyze a distribution dual to (1) and (2). It will be referred to as the multitime Wigner higher order distribution (MTWHOD); it is defined by

$$W_k(\omega,t_1,t_2,\ldots,t_k) = \frac{1}{(2\pi)^k}\int_{\theta_1}\int_{\theta_2}\cdots\int_{\theta_k} X^*(\omega-A) \times \prod_{i=1}^{L-1} X^*(\omega-A+\theta_i) \times \prod_{i=L}^{k} X(\omega-A+\theta_i)\prod_{i=1}^{k} e^{jt_i\theta_i}d\theta_i$$
$$\text{with } A = \frac{1}{k+1}\sum_{i=1}^{k}\theta_i \quad (3)$$

and $1 \le L \le k$. Equation (3) is dual to (1) when $L = 1$. The MTWHOD in terms of $x(t)$ dual to (2) is

$$W_k(\omega,t_1,t_2,\ldots,t_k) = \int_\tau x^*\left(\sum_{i=1}^{k}t_i + \frac{\tau}{k+1}\right) \times \prod_{i=1}^{L-1} x^*\left(-t_i + \frac{\tau}{k+1}\right) \times \prod_{i=L}^{k} x\left(t_i - \frac{\tau}{k+1}\right)e^{j\tau\omega}d\tau. \quad (4)$$

All properties of the MTWHOD are just dual to the ones described in [2] and [3] for the Wigner higher order spectra. Our intention is to analyze the MTWHOD in the case of multicomponent signals. This analysis will serve as a basis for the derivation of a distribution with interesting properties.

II. Autoterms and Crossterms in the MTWHOD

Consider a multicomponent signal formed as a sum of short duration (pulse) signals

$$x(t) = \sum_{m=1}^{M} x_m(t-d_m), \quad (5)$$

where $x_m(t)\,(m = 1,2,\ldots,M)$ are such that $x_m(t) = 0$ for $|t| \succeq \epsilon$, with ϵ being small as compared with the considered time interval.

The integrand in (4) is different from zero only if the following inequalities are satisfied:

$$\left|\frac{\tau}{k+1} + \sum_{q=1}^{k}t_q - d_i\right| < \epsilon \quad \wedge \left|-t_m + \frac{\tau}{k+1} - d_{j_m}\right| < \epsilon \quad \wedge \left|t_n - \frac{\tau}{k+1} - d_{l_n}\right| < \epsilon \quad (6)$$

where $i, j_m, l_n = 1,2,\ldots,M$; $m = 1,2,\ldots,L-1$; $n = L, L+1,\ldots,k$.

In order to analyze the autoterms, consider the case $M = 1$ with $d_1 \equiv d$, when the crossterms do not exist. Eliminating $t_q\,(q = 1,\ldots,k)$ from the first inequality in (6) and τ from

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the remaining ones, we get

$$\left|\frac{\tau}{k+1}-\frac{(2L-k-1)d}{k+1}\right|<\epsilon$$
$$\wedge\left|-t_m+\frac{(2L-k-1)d}{k+1}-d\right|<2r\epsilon$$
$$\wedge\left|t_n-\frac{(2L-k-1)d}{k+1}-d\right|<2r\epsilon \quad (7)$$

where $r = k/(k+1)$ and m, n are indexes as in (6).

From the first inequality in (7), we see that the location of the autoterms along τ depends on the signal's position d for any L except for $L = (k+1)/2$. This case was preferred in [1] in cumulant analysis. When $L = (k+1)/2$, the autoterms are located at the τ axis origin and its vicinity. From the remaining inequalities in (7), one may conclude that for $L = (k+1)/2$ the autoterms lie, in the k-dimensional $t_1, t_2, \ldots, t_k$ space, along line s defined by

$$s: t_1=-t, t_2=-t,\ldots,t_{L-1}=-t, t_L=t, t_{L+1}$$
$$=t,\ldots,t_k=t \text{ at the points } t=d. \quad (8)$$

The illustration of MTWHOD of the second order ((4) with $k = 2$ dual to the Wigner bispectrum) that cannot satisfy the condition $L = (k+1)/2$, as well as the illustration of MTWHOD of the third order ((4) with $k = 3$ dual to the Wigner trispectrum), as the lowest one satisfying the previous condition (if one does not count the well-known Wigner distribution), are given in Figs. 1(a), and (b), respectively.

If $M > 1$, then for $L = (k+1)/2$, considering only line s, the regions where the integrand in (4) is different from zero may be obtained from (6) with $j_m = l_n = j$:

$$\left|\frac{\tau}{k+1}-\frac{d_i-d_j}{k+1}\right|<\epsilon\wedge\left|-t_m+\frac{d_i-(k+2)d_j}{k+1}\right|<2r\epsilon$$
$$\wedge\left|t_n-\frac{d_i+kd_j}{k+1}\right|<2r\epsilon. \quad (9)$$

It follows from (9) that the components of the integrand in (4), corresponding to the crossterms, are dislocated from the τ axis origin. They lie around $\tau = d_i - d_j, i, j = 1, 2, \ldots, M$ and $i \neq j$. From (9), one may conclude that the integration over autoterms $(i = j)$ is completely performed, and at the same time, the crossterms on s are removed if we use the window $w_L(t)$ in (4) of width $2T_m (w_L(\tau) = 0$ for $|\tau| > T_m)$:

$$(k+1)\epsilon \le T_m < \min_{i,j}|d_i-d_j| - (k+1)\epsilon. \quad (10)$$

III. L-WIGNER DISTRIBUTION

The MTWHOD, which is given by (4), with $L = (k+1)/2$ along line s (defined by (8)) has the form

$$\mathrm{LWD}_L(\omega,t)=\int_\tau x^{*L}\Big(t-\frac{\tau}{2L}\Big)x^L\Big(t+\frac{\tau}{2L}\Big)w_L(\tau)e^{-j\omega\tau}d\tau. \quad (11)$$

This distribution will be called the L-Wigner distribution. For $L = 1$, it is reduced to the Wigner distribution.[1]

[1]We have initially defined the L-Wigner distribution in an intuitive way, analyzing the instantaneous frequency representation of monocomponent signals by the common time-frequency distributions in [5]. The distributions described in [6] and [7], resulting from the analysis of Wigner trispectrum, may be treated as the special cases of the L-Wigner distribution with $L = 2$ as well.

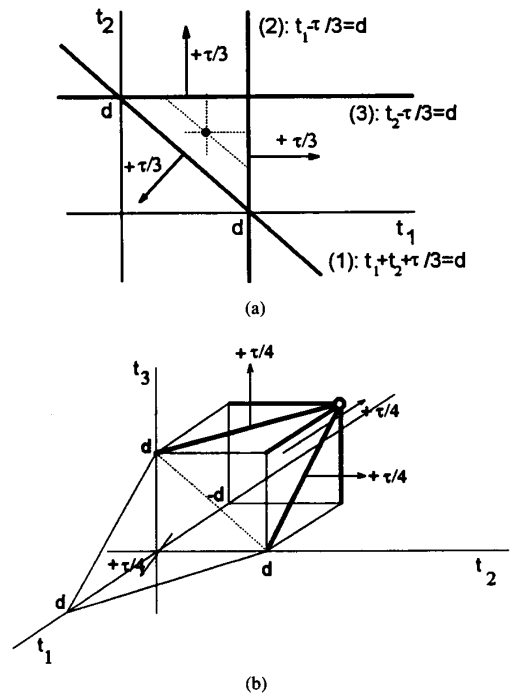

Fig. 1. Illustration of the MTWHOD of (a) second order and (b) third order.

Consider now multicomponent signal, which is formed as a sum of long duration signals:

$$x(t)=\sum_{m=1}^{M} r_m(t)e^{j\phi_m(t)} \quad (12)$$

where the amplitudes $r_m(t)$ are slow varying, i.e., such that $r_m(t)$ may be treated as a constant inside the window $w(\tau); r_m(t \pm \tau)w(\tau) \cong r_m(t)w(\tau)$.

The L-Wigner distribution of signal (12) is

$$\mathrm{LWD}_L(\omega,t)=\sum_{m=1}^{M}\int_\tau r_m^L\Big(t-\frac{\tau}{2L}\Big)r_m^L\Big(t+\frac{\tau}{2L}\Big)$$
$$\times e^{jL\phi_m(t+\tau/2L)-jL\phi_m(t-\tau/2L)}w_L(\tau)e^{-j\omega\tau}d\tau$$
$$+\mathrm{CT}(\omega,t) \quad (13)$$

where $\mathrm{CT}(\omega, t)$ denotes the crossterms along the frequency axis. Although the total number of crossterms in the L-Wigner distribution was drastically reduced when compared with the MTWHOD (see (4)), it still remains very large.[2] Our aim is not the analytical treatment of crossterms but the definition of a distribution for time-frequency analysis that will be, under certain conditions, crossterm free. On the basis of the crossterm

[2]The total number of terms (autoterms plus crossterms) in (4) is $N = M^{k+1}$ for $k > 1$ and $N = M(M+1)/2$ for $k = 1$. The maximal number of terms in the L-Wigner distribution may be obtained from the recursive formula $N_b = N_{b-1}(N_{b-1}+1)/2$. The starting value is $N_o = M$, and the final one is $N = N_m$ for $L = 2^{m-1}$. This form of L is used in the realization; see Section IV. For example, taking $M = 4$ and $k = 3\,(L = 2)$, we get, for MTWHOD, $N = 256$, whereas for the L-Wigner distribution, $N = 55$.

locations, we have shown in Section II that these may be eliminated in the case of nontime-overlapping pulses. In the section that follows, we will show that crossterm elimination is possible in the case of nonfrequency-overlapping long signals (tones) as well.

Expanding $\phi_m(t \pm \tau/2L)$ into a Taylor series around t up to the third order term, we get

$$\mathrm{LWD}_L(\omega,t) = \frac{1}{2\pi}\sum_{m=1}^{M} r_m^{2L}(t)\delta(\omega - \phi_m'(t)) \otimes \mathrm{FT}\left\{ e^{j\frac{\phi_m^{(3)}(t+\tau_1)+\phi_m^{(3)}(t-\tau_2)}{3!L^2}(\tau/2)^3} \right\} \otimes W_L(\omega) + \mathrm{CT}(\omega,t) \quad (14)$$

where $\otimes$ denotes the frequency domain convolution $0 \le |\tau_1|, |\tau_2| \le |\frac{\tau}{2L}|$, and $\mathrm{FT}\{O\}$ is the FT operator.

From (14), one may conclude that the generalized power $r_m^{2L}(t)$ is concentrated at the instantaneous frequencies $\phi_m'(t)$. The distortions caused by the shape of the phase function are due to the existence of its third and higher order derivatives. If the instantaneous frequency is a linear function of time, then the Wigner distribution ($L = 1$) produces the ideal concentration. However, if that is not the case, then $L > 1$ dramatically reduces the distortion. In other words, the L-Wigner distribution locally linearizes the instantaneous frequency function (some interesting results dealing with the polynomial phase function are reported in [8] and [9]).

IV. A Method for the L-Wigner Distribution Realization

The realization of L-Wigner distribution may be efficiently done using the recursive formula

$$\mathrm{LWD}_{2L}(\omega,t) = \frac{1}{\pi}\int_\theta \mathrm{LWD}_L(\omega+\theta,t)\mathrm{LWD}_L(\omega-\theta,t)d\theta. \quad (15)$$

Note that $\mathrm{LWD}_{2L}(\omega,t)$ will be crossterm free provided that 1) the starting transform is crossterm free, and 2) the recursions do not introduce crossterms. We will show that these requirements may be met under certain conditions. Let us prove that assumption 2) may hold. Suppose that $\mathrm{LWD}_L(\omega,t)$ in (15) is crossterm free. Its autoterms are located around $\phi_m'(t), m = 1,2,\ldots,M$; see (14). The terms in $\mathrm{LWD}_{2L}(\omega,t)$ are located along the θ axis at $|\theta - [\phi_i'(t) - \phi_j'(t)]/2| < W_{wL}$, where $2W_{wL}$ is the width of $W_L(\omega)$ in (14). If the integration in (15) is performed using the frequency domain window $P(\theta)$, then the crossterms in $\mathrm{LWD}_{2L}(\omega,t)$ may be suppressed, whereas the integration over autocomponents ($i = j$) is completed. This is possible, provided that the width $2W_p$ of window $P(\theta)\,(P(\theta) = 0$ for $|\theta| > W_p)$ satisfies

$$W_{wL} \le W_p < \min_{i,j}|\phi_i'(t) - \phi_j'(t)|/2 - W_{wL}. \quad (16)$$

The modified L-Wigner distribution is

$$\mathrm{MLWD}_{2L}(\omega,t) = \frac{1}{\pi}\int_\theta P(\theta)\,\mathrm{LWD}_L(\omega+\theta,t)\mathrm{LWD}_L(\omega-\theta,t)d\theta. \quad (17)$$

The distortion due to the higher order-phase derivatives in (14) is neglected in this analysis.

The starting iteration ($2L = 1$) is

$$\mathrm{MWD}(\omega,t) = \frac{1}{\pi}\int_\theta P(\theta)\mathrm{STFT}(\omega+\theta,t)\mathrm{STFT}^*(\omega-\theta,t)d\theta \quad (18)$$

where $\mathrm{STFT}(\omega,t)$ is the short-time Fourier transform defined as $\mathrm{STFT}(\omega,t) = \mathrm{FT}_\tau\{x(t+\tau)w(\tau)\}$.

This way, the resulting modified L-Wigner distribution is crossterms free (terms denoted by $\mathrm{CT}(\omega,t)$ in (13) do not exist) if the starting transform $\mathrm{STFT}(\omega,t)$ is crossterm free (which is the case if the signal components do not overlap in the time-frequency plane) and if, at the same time, (16) is satisfied in each iteration. Note that if (16) cannot be satisfied for some i, j, and t (i.e., $|\phi_i'(t) - \phi_j'(t)|/2 - W_{wL} < W_{wL}$), then the crossterms will appear at that instant t between the ith and jth signal components.

The discrete forms of (17) and (18), with a rectangular window $P(\theta)$, are

$$\mathrm{MWD}(n,k) = |\mathrm{STFT}(n,k)|^2 + 2\sum_{i=1}^{N_p}\mathrm{Re}\{\mathrm{STFT}(n,k+i)\mathrm{STFT}^*(n,k-i)\}$$
$$\mathrm{MLWD}_{2L}(n,k) = \mathrm{MLWD}_L^2(n,k) + 2\sum_{i=1}^{N_p}\mathrm{MLWD}_L(n,k+i)\mathrm{MLWD}_L(n,k-i)\} \quad (19)$$

where $2N_p + 1$ is the width of the discrete form of $P(\theta)$. In [4], it is shown that the realization of $\mathrm{MWD}(n,k)$ may be computationally very efficient. Here, we will only indicate that the oversampling in the modified Winger distribution (18) is not necessary because the aliasing components are removed in the same way as the crossterms are[3] [4], [10]. The same conclusions are valid for the L-Wigner distribution.

V. Numerical Example

As a numerical example, consider the multicomponent signal

$$x(t) = e^{j8\pi(2-t)^2} + e^{j12\sin[3\pi/2(t+1)]-j20\pi t} + 2e^{-[25(t+0.4)]^2+j4\pi t} + 2e^{-[25(t-0.3)]^2-j40\pi t} + n(t).$$

The first two components are of form (12), whereas the second two are of form (5). The last component $n(t)$ is a Gaussian white noise with the variance $\sigma_n = 0.3$. The spectrogram of $x(t)$ is given in Fig. 2(a). In the STFT calculation, a window $w(\tau) = h^2(\tau)$ (where $h(\tau)$ is a Hanning window) of the width $2T_m = 1$ is used (the window selection in the STFT is very widely studied). The number of samples

[3] A sampled signal has a periodic spectrum. It may be formally treated as a multicomponent signal with an infinite number of components equally spaced with the distance $\omega_p = 2\pi/\Delta t$, where Δt is the sampling interval. Note that the distance ω_p, which is considered from the point of view of (16), is usually less demanding than the minimal distance between the autocomponents in the original multicomponent signal.

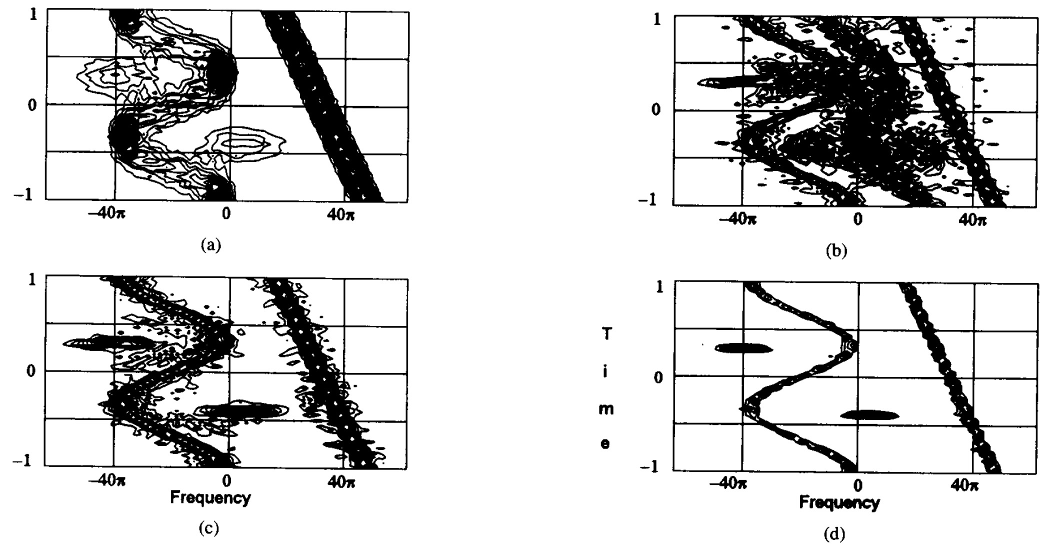

Fig. 2. Time-frequency representation of a multicomponent signal: (a) Spectrogram, (b) Wigner distribution; (c) modified Wigner distribution, (d) modified L-Wigner distribution with $L = 4$. Sampling interval $\Delta t = 1/64$, rectangular window $P(\theta)$, and Hanning squared window $w(\tau)$ are used.

is $N = 64$. Note that the number of nonzero samples in the Fourier transform of $w(\tau)$ is $N = 2N_w + 1 = 5$, with $N_w = 2$ and $W_w = 4\pi$. The Wigner distribution, which is calculated by the standard routines [4], is shown in Fig. 2(b). The modified Wigner distribution, which was obtained from the STFT and (19), is presented in Fig. 2(c). A rectangular window $P(\theta)$, whose width in the discrete domain is defined by $N_p = 3$, is used in (19). In order to ensure the integration over autocomponents ($W_{wL} \leq W_p$ in (16)), a very narrow window $P(\theta)$ is usually sufficient. In our example, with the described window $w(\tau), N_p = N_w = 2$ is sufficient, but we will take a margin with $N_p = 3$ because the distortion due to the higher order phase derivatives is not included in (16). The value $N_p = 3$ i.e., $W_p = 6\pi$ in the analog domain, provides the complete elimination of crossterms (as well as the aliasing effects) between the components whose instantaneous frequencies are more than 20π apart along the frequency axis. This holds for all considered components in the example (otherwise, the crossterms would appear but still in a very reduced form). The MTWHOD along line s, i.e., the modified L-Wigner distribution with $L = 4$, calculated with the same window $P(\theta)$ and using (19) in two iterations ($L = 2$ and $L = 4$) is given in Fig. 2(d). The analysis of the crossterm elimination is the same as in case of the modified Wigner distribution.

VI. Conclusion

An efficient distribution (L-Wigner distribution) for time-frequency analysis is derived from the dual definition of the Wigner higher order spectra. This distribution has the following advantages over the standard Wigner distribution: There is a very high distribution concentration at the instantaneous frequency, crossterms are removed (or reduced), and signal oversampling is not necessary

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