L-Class of Time-Frequency Distributions

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Abstract—**The L-class of distributions for time-frequency signal analysis is derived and presented, generalizing the recently defined L-Wigner distribution. Some particular distributions belonging to this class are introduced.**

I. Introduction

THE oldest method for time-frequency signal analysis is based on the short time Fourier transform (STFT). It is a linear signal transformation. Many performances of the signal's representation may be improved using the quadratic distributions ([1], [2] and numerous references therein). The first quadratic representation was based on the Wigner distribution (WD). Afterward, many other quadratic distributions have been defined. Cohen has given the general form for the shift covariant time-frequency distributions [1]. Analyzing the instantaneous frequency presentation in [9], the L-Wigner distribution (L-WD), as the higher order generalization of the WD, is proposed and presented.

In the last few years, higher order time-varying spectra (HOTVS) have become hot topics in the time-frequency analysis [4]–[6], [8], [10], [11]. It turns out that the various reduced forms of the HOTVS [5], [6] are the special cases of the L-WD. It is shown that the L-WD (in its dual form) is optimal in the analysis of multicomponent signals using the Wigner higher order spectra [8], [11]. The L-WD is derived as an optimal one in the analysis of multicomponent signals using the multitime Wigner higher order distributions [10]. Combined with the method for time-frequency analysis presented in [7], the L-WD produced a powerful tool for time-frequency analysis [8], [10]–[12], [14]. The L-WD has been defined in the case of multidimensional signals as well [13]. The extension of the L-WD to the L-class of distributions (L-CD) is done in this letter.

II. Definition of the L-Class of Distributions

The L-WD, in its pseudo form, is defined as [9], [10], [12], [14]:

$$LWD_L(t,\omega)=\int_\tau w_L(\tau)x^{*L}(t-\frac{\tau}{2L})x^L(t+\frac{\tau}{2L})e^{-j\omega\tau}d\tau. \quad (1)$$

The word "pseudo" will be used to indicate when the window $w_L(\tau)$ is included. It is known that the ambiguity function may be defined as a 2-D Fourier transform (FT_{2D}) of the WD [1], [2], [15]. Here, we will introduce the L-ambiguity function and use it to define the L-generalized characteristic function and the L-CD.

Definition 1: The L-ambiguity function is a FT_{2D} of the L-Wigner distribution:

$$LA_L(\theta,\tau)=\int_u x^{*L}(u-\frac{\tau}{2L})x^L(u+\frac{\tau}{2L})e^{-j\theta u}du. \quad (2)$$

Definition 2: A product of $LA_L(\theta,\tau)$ and an arbitrary function $c_L(\theta,\tau)$ called the kernel produces the L-generalized characteristic function:

$$MG_L(\theta,\tau)=c_L(\theta,\tau)\int_u x^{*L}(u-\frac{\tau}{2L})x^L(u+\frac{\tau}{2L})e^{-j\theta u}du. \quad (3)$$

Definition 3: The L-class of distributions is an inverse FT_{2D} of the L-generalized characteristic function:

$$LD_L(t,\omega)=\frac{1}{2\pi}\int_\theta\int_u\int_\tau c_L(\theta,\tau)x^{*L}(u-\frac{\tau}{2L})\cdot x^L(u+\frac{\tau}{2L})e^{-j\omega\tau}e^{-j\theta(u-t)}dud\theta d\tau. \quad (4)$$

For $L=1$, this class of distributions reduces to the Cohen class [1].

Distribution (4) may be understood as an inverse FT_{2D} of the product of $LA_L(\theta,\tau)$ and $c_L(\theta,\tau)$. Thus, it is equal to the 2-D convolution of $\Pi_L(t,\omega)=FT_{2D}\{c_L(\theta,\tau)\}$ and $LWD_L(t,\omega)$:

$$LD_L(t,\omega)=\frac{1}{2\pi}\int_u\int_v \Pi_L(t-u,\omega-v)LWD_L(u,v)dudv. \quad (5)$$

All distributions from the L-CD may be treated as the smoothed versions of the L-WD.[1]

[1] Expressions (4) and (5) may be extended to the time-scale distributions [3]

$$LAD_L(t,a)=\frac{1}{2\pi}\int_\theta\int_u\int_\tau c_L(a\theta,\tau/a)x^{*L}(u-\frac{\tau}{2L})\cdot x^L(u+\frac{\tau}{2L})e^{-j\omega_0\tau/a}e^{-j\theta(u-t)}dud\theta d\tau.$$

$$LAD_L(t,a)=\frac{1}{2\pi}\int_u\int_v \Pi_L(\frac{t-u}{a},\omega_0-av)LWD_L(u,v)dudv$$

where a is a scale factor $a\equiv\omega_o/\omega$, and ω_0 is a constant. The properties and spacial cases of this class of distributions may be derived starting from the results given in this paper and the ones presented in [3] and [12].

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III. Some General Properties

In this section, we will list some basic properties of the distributions belonging to the L-CD. Many of them may be obtained in a straightforward manner, generalizing the ones of the Cohen class [1], [2]. These properties will be given without proofs or any additional explanation. Attention will be paid only to those for which the L-CD behaves in a qualitatively different manner than the Cohen class.

1) A distribution from the L-CD is real if its L-generalized autocorrelation function

$$LRA_L(t,\tau) = \frac{1}{2\pi}\int_\theta\int_u c_L(\theta,\tau)x^{*L}(u-\frac{\tau}{2L}) \cdot x^L(u+\frac{\tau}{2L})e^{-j\theta(u-t)}dud\theta \tag{6}$$

is Hermitian $LRA_L(t,\tau) = LRA_L^*(t,-\tau)$. This condition is satisfied for $c_L(\theta,\tau) = c_L^*(-\theta,-\tau)$.

2) The L-CD is time- and frequency-shift invariant if $c_L(\theta,\tau)$ is not time- (t) and frequency- (ω) dependent.

3) If a signal is time limited to $|t| < T$, then $LD_L(t,\omega)$ is limited to the same time interval if $C_L(t,\tau) = FT_\theta\{c_L(\theta,\tau)\} = 0$ for $|t/\tau| > 1/(2L)$.

4) If a signal is band limited to $|\omega| < \omega_m$, then the L-CD is band limited to the same bandwidth if $\mathbf{C}_L(\theta,\omega) = FT_\tau\{c_L(\theta,\tau)\} = 0$ for $|\omega/\theta| > 1/(2L)$.

5) If the distribution $LD_L(t,\omega)$ corresponds to $x(t)$, then $LD_L(at,\omega/a)$ is the distribution of $|a|^{\frac{1}{2L}}x(at)$, provided that $c_L(\theta/a, a\tau) = c_L(\theta,\tau)$.

6) The integral of $LD_L(t,\omega)$ over ω is equal to the generalized power $|x(t)|^{2L}$, if $c_L(\theta,0) = 1$.

7) If $c_L(0,0) = 1$, then

$$\frac{1}{2\pi}\int_t\int_\omega LD_L(t,\omega)dtd\omega = \int_t |x(t)|^{2L}dt = \|x(t)\|_{2L}^{2L}$$

where $\|x(t)\|_{2L}^{2L}$ is the $2L$th power of a $2L$th norm of the signal $x(t)$ (generalized energy).

8) If $c_L(0,\tau) = 1$, then the integral of $LD_L(t,\omega)$ over time is shown in the equation at the bottom of the page, where $X_L(\omega)$ is the Fourier transform of $x^L(t)$.

9) The frequency domain form of $LD_L(t,\omega)$ is

$$LD_L(t,\omega) = \frac{L}{4\pi^2}\int_\theta\int_u\int_\tau c_L(\theta,\tau)X_L^*(Lu-\frac{\theta}{2}) \cdot X_L(Lu+\frac{\theta}{2})e^{j\theta t}e^{-j\tau(\omega-u)}dud\theta d\tau.$$

10) For the signal $x(t) = A(t)\exp(j\phi(t))$, the mean frequency $\langle\omega\rangle_t = \int_\omega \omega LD_L(t,\omega)d\omega / \int_\omega LD_L(t,\omega)d\omega$ is equal to the instantaneous frequency (IF) $\phi'(t)$, if $c_L(\theta,0) = 1$ and $\frac{\partial c_L(\theta,\tau)}{\partial\tau}_{|\tau=0} = 0$.

11) *Frequency-Modulated Signals Representation:* The ideal distribution, concentrated along the IF, is defined by $2\pi A^2\delta(\omega - \phi'(t))$ or by $A^2W(\omega - \phi'(t))$ if a finite time interval, determined by the window $w(\tau) = FT^{-1}\{W(\omega)\}$, is used. For the signal $x(t) = Ae^{j\phi(t)}$, this form may be obtained in the Cohen class of distributions, only if the IF is a linear function $\phi'(t) = at + b$. The distribution that produces this concentration is the WD (or the pseudo WD) [9], [12], [14]. If the IF variations are of a higher order than linear, then no distribution (with signal-independent kernel) from the Cohen class can produce the ideal concentration.

Theorem 1: The L-class of distributions, for $L \to \infty$, is equal to the ideal form $A^{2L}W(\omega - \phi'(t))$ for any frequency-modulated signal $x(t) = Ae^{j\phi(t)}$ if the derivatives of the phase function $\phi(t)$ are finite and if $\lim_{L\to\infty} c_L(\theta,\tau) = w(\tau)$, where $w(\tau)$ is a finite duration window $w(\tau) = FT^{-1}\{W(\omega)\}$.

Proof: For a signal of the form $x(t) = Ae^{j\phi(t)}$, expanding $\phi(u \pm \tau/2L)$ into a Taylor series around u, up to the third order term, we get

$$LD_L(t,\omega) = \frac{1}{2\pi}A^{2L}\iiint_{-\infty}^{\infty} c_L(\theta,\tau) \cdot e^{j\phi'(u)\tau}e^{j\frac{\phi^{(3)}(u+\tau_1)+\phi^{(3)}(u-\tau_2)}{3!L^2}\left(\frac{\tau}{2}\right)^3}e^{j\theta t - j\omega\tau - j\theta u}dud\theta d\tau \tag{7}$$

where τ_1, τ_2 are variables $0 \le |\tau_{1,2}| \le \left|\frac{\tau}{2L}\right|$ If $\phi^{(3)}(\tau)$ and $\phi^{(n)}(\tau)$, $n > 3$ are finite and the variable τ may assume only finite values, then for a large L, the value $\lim_{L\to\infty}\left[\exp(j\frac{\phi^{(3)}(u+\tau_1)+\phi^{(3)}(u-\tau_2)}{3!L^2}\frac{\tau^3}{8})\right] = 1$ (since the convergence is of order $1/L^2$, it turned out [8]–[12], [14] that it is practically true for $L \ge 4$); therefore, we get

$$LD_L(t,\omega) \cong \frac{1}{2\pi}A^{2L}\iiint_{-\infty}^{\infty} \cdot c_L(\theta,\tau)e^{j\phi'(u)\tau}e^{j\theta t - j\omega\tau - j\theta u}dud\theta d\tau. \tag{8}$$

Relation (8) may be written in the form

$$LD_L(t,\omega) \cong A^{2L}\int_{-\infty}^{\infty}\Pi_L(t-u,\omega-\phi'(u))du \tag{9}$$

where $\Pi_L(t,\omega)$ is the FT_{2D} of $c_L(\theta,\tau)$. If $\lim_{L\to\infty} c_L(\theta,\tau) = w(\tau)$, then $\Pi_L(t,\omega) = \delta(t)W(\omega)$, and $LD_L(t,\omega) = A^{2L}W(\omega - \phi'(t))$. This form corresponds to the ideal distribution. Q.E.D.

$$\int_t LD_L(t,\omega)dt = L|X_L(L\omega)|^2 = \left|\int_{\theta_1}\int_{\theta_2}\cdots\int_{\theta_{L-1}} X(\omega L - \theta_1 - \cdots - \theta_{L-1})X(\theta_1)X(\theta_2)\cdots X(\theta_{L-1})\frac{d\theta_1\cdots d\theta_{L-1}}{(2\pi)^{L-1}}\right|^2$$

12) *Theorem 2:* An Lth order distribution, belonging to the L-class of distributions, may be obtained from its $L/2$th order form if $c_L(\theta,\tau) = c_{L/2}(u,\tau/2)c_{L/2}(\theta-u,\tau/2)$ for any u.

Proof: It is evident from (2) that

$$LA_L(\theta,\tau) = LA_{L/2}(\theta,\tau/2) *_\theta LA_{L/2}(\theta,\tau/2)$$

where $*_\theta$ is a convolution in θ. According to the theorem's assumption, it follows that

$$MG_L(\theta,\tau) = MG_{L/2}(\theta,\tau/2) *_\theta MG_{L/2}(\theta,\tau/2).$$

Taking the FT_{2D} of both sides, we get

$$LD_L(t,\omega) = \frac{1}{\pi}\int_\lambda LD_{L/2}(t,\omega+\lambda)LD_{L/2}(t,\omega-\lambda)d\lambda. \quad (10)$$

Q.E.D.

Note: The preposition of Theorem 2 is satisfied by the Wigner, Rihaczek, Page, Levin, etc., type kernels [1], [2].

IV. Some Specific Distributions from the L-CD

Here, we will define some particular distributions belonging to the L-CD. A few interesting properties will be considered for each of them.

A. L-Wigner Distribution

We have already given the definition of the pseudo L-WD (1), which is the most important member of this class. Since it is taken as a basis for the generalization, obviously, its kernel is $c_L(\theta,\tau)=1$, or for its pseudo form $c_L(\theta,\tau) = w_L(\tau)$. The properties, various derivations, realizations and applications of the pseudo L-WD are studied in detail in [8]–[14].

B. L-Rihaczek Distribution

The L-class counterpart of the Rihaczek distribution, in the pseudo form is defined as

$$LRD_L(t,\omega) = \int_\tau x^L(t+\frac{\tau}{L})x^{*L}(t)w_L(\tau)e^{-j\omega\tau}d\tau. \quad (11)$$

This distribution is obtained from the general one with $c_L(\theta,\tau) = e^{j\theta\tau/2L}w_L(\tau)$.

For a frequency-modulated signal $x(t) = A\exp(j\phi(t))$, with $\phi(t) = a+bt+ct^2/2$, after expansion of $\phi(t+\frac{\tau}{L})$ into a Taylor series, we get

$$LRD_L(t,\omega) = A^{2L}\delta(\omega-\phi'(t))*_\omega \cdot FT\Big\{w_L(\tau)e^{jc\tau^2/(2L)}\Big\} \underset{L\to\infty}{\rightarrow} A^{2L}W(\omega-\phi'(t)).$$

This could be expected since the kernel $c_L(\theta,\tau) = e^{j\theta\tau/2L}w_L(\tau) \to w(\tau)$ as $L\to\infty$. However, the convergence in this case is of order $1/L$, which is worse than in the pseudo L-WD.

C. L-Spectrogram and L-Short Time Fourier Transform

The L-spectrogram is defined as the squared modulus of the L-short time Fourier transform (L-STFT)

$$LSPEC_L(t,\omega) = |LSTFT_L(t,\omega)|^2 = \left|\int_\tau w_L(\tau)x^L(t+\frac{\tau}{L})e^{-j\omega\tau}d\tau\right|^2. \quad (12)$$

Many specific properties of the L-STFT and the L-spectrogram may be easily derived from the widely known properties of the STFT. Here, we will focus only on the one that treats the dependence of frequency and time resolution on the window function. First, assume that the signal $x(t)$ is short and concentrated at $t=0$ into an interval $\Delta t \to 0$. If the window $w_L(t)$ is time limited to $|t| < T/2$ (where $T \gg \Delta t$), then the L-STFT is time limited to $|t| < T/(2L)$, i.e., its duration is $d = T/L$. If we now assume a sinusoidal signal $x(t) = \exp(j\omega_0 t)$ and the same window, we get $LSTFT_L(\omega,t) = W_L(\omega-\omega_0)e^{jL\omega_0 t}$. For example, let the window be rectangular. The width of its Fourier transform $W_L(\omega)$ (the width of its main lobe) is $D = 4\pi/T$. The product of the durations d and D (the form of uncertainty principle in this case) is $dD = 4\pi/L$. This relation states that the L-STFT, with a given L, cannot be localized in the time-frequency plane with arbitrary small d and D simultaneously (representing the resolutions in time and frequency directions). However, the previous relation permits us to draw an important conclusion: By increasing L, the product dD can be made arbitrary small, meaning arbitrary high resolutions simultaneously in both directions.

D. L-Reduced Interference Distributions (L-RID)

Although the WD satisfies most of the desired properties [1], [2], it is rarely used in its original form. The main reason lies in the very emphatic cross-term effects. These effects may be even more emphasized in the L-WD for $L > 1$ since the Lth power of signal may increase the number of cross terms.[2] Unfortunately, these terms behave as the regular auto terms. Thus, the straightforward generalization of the RID distributions (Choi-Williams, Zao-Atlas-Marks, Born-Jordan, Sinc, etc., [1], [2]) would reduce only a limited number of cross terms resulting from the product of $x^L(t+\tau/2L)$ and $x^{*L}(t-\tau/2L)$.

However, *all cross terms may be efficiently reduced or removed using recursive formula (10).* Starting from the distribution that is cross-terms free, we may control the cross-terms appearance in the subsequent iterations using the function

[2] The L-Wigner distribution reduces the possibility of cross terms appearing between two time separated components. Suppose that there are two signals such that one exists for $|t-t_1| < T_1$ and the other for $|t-t_2| < T_2$. The cross term in (1) is located at $\left|t - \frac{t_1+t_2}{2}\right| < \frac{T_1+T_2}{2}$ and $\left|\frac{\tau}{L} - \frac{t_1-t_2}{2}\right| < \frac{T_1+T_2}{2}$. Keeping the window $w_L(\tau)$ width unchanged, the possibility of satisfying the second inequality is significantly reduced as L increases. For $L\to\infty$, there is no cross term between time-separated components.

$P(\lambda)$ that is of lowpass filter type:

$$LDM_L(t,\omega) = \frac{1}{\pi}\int_\lambda P(\lambda) \cdot LD_{L/2}(t,\omega+\lambda)LD_{L/2}(t,\omega-\lambda)d\lambda. \quad (13)$$

The numerical aspects of realization of distributions, which may be written in form (13), are described in [7] and [10].

V. On the Realization

In our previous work, we described two methods for the L-WD realization. They can be directly applied to any distribution from the L-CD. 1) The **direct method** is based on the signal raising to the Lth power, its oversampling L times, and keeping unchanged the number of samples used for calculation. Regarding the last assumption, this method is not computationally much more demanding than the realization of any ordinary ($L = 1$) distribution, but besides the oversampling, its disadvantage is in increasing the number of cross terms for multicomponent signals that do overlap in time. This deficiency is overcome by 2) the **recursive method**, which is based on relation (13). This method provides the following advantages: The cross terms are reduced (eliminated); oversampling is not necessary; and computationally, it may be more efficient than the direct method. The particular numerical examples realized by these methods, along with the details on the methods, may be found in [7]–[14].

VI. Conclusion

The L-class of distribution, as a generalization of the Cohen class, is presented. The L-distributions, corresponding to the well-known distributions for time-frequency analysis, are derived.

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