Highly Concentrated Time-Frequency Distributions: Pseudo Quantum Signal Representation

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***Abstract*— Distributions that are highly concentrated in the time-frequency plane are presented. Since the idea for these distributions originated from the Wigner representation in the quantum mechanics, a review of this representation is done in the first part of the paper. Abstracting the physical sense of the quantum mechanics representation, we defined the "pseudo quantum" signal representation. On the basis of a signal, the "pseudo wave function" with the corresponding "pseudo particle" having the "pseudo momentum" $\wp = \hbar_f \omega$ is generated. By varying the value of $\hbar_f$, one is in a position to influence the concentration of the "pseudo quantum" (time-"pseudo momentum") signal's presentation while keeping its most important time-frequency properties invariant. From this reflection, an efficient distribution for the time-frequency (time-"pseudo momentum") signal analysis is obtained. This distribution produces as high a concentration in the time-frequency (time-"pseudo momentum") plane as the L-Wigner distribution; however, it may satisfy the marginal properties. The theory is illustrated with examples.**

I. Introduction

CONCENTRATION of a time-frequency distribution is one of its very important and intensively studied properties [1], [6], [7], [9], [11], [20]. The Wigner distribution, which is defined in quantum mechanics [3] and introduced in signal analysis by Ville [5], is the only one from the Cohen class of distributions [1], [2] (with signal independent kernel) that may produce the complete distribution concentration along the instantaneous frequency when it is a linear function of time [6], [7]. Concentration improvement, in the case of nonlinear instantaneous frequency, may be achieved using higher order distributions, like, for example, the polynomial Wigner distributions [9], [10] or the L-Wigner distributions [7], [8], [11], [12]. Higher order distributions do not satisfy common marginal properties [1], [2], but rather their generalized forms. In this paper, on the basis of quantum mechanics form of the Wigner distribution, a highly concentrated time-frequency distribution will be proposed. It will be concentrated as high as the L-Wigner distribution, but in addition, it will satisfy unbiased energy condition, time marginal and, for asymptotic signals, frequency marginal properties.

The paper is organized as follows. In the first part of the paper, a review of the quantum mechanics Wigner representation is provided[1] since the highly concentrated distribution is derived from its analysis. Distribution definition and its properties are presented next. Finally, a method for the realization of the highly concentrated distribution, in the case of multicomponent signals (including noisy ones and the ones that intersect in the time-frequency plane), is presented.

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II. Review of the Quantum Mechanics Wigner Representation

Classical equations describing a particle motion are given by $dx/dt = p/m$ and $dp/dt = -\nabla V(x) = -V'(x)$, where

x	position,
m	mass,
$p = mv$	momentum of a particle,
$V(x)$	potential.

If the initial conditions are not given by $x(0)$ and $p(0)$, but rather by their probability distribution $W(p, x, t = 0)$; then, the particle dynamics is described by Liouville's equation:

$$\frac{\partial W}{\partial t} + \frac{p}{m}\frac{\partial W}{\partial x} = V'(x)\frac{\partial W}{\partial p}. \tag{1}$$

Quantum mechanics generalization of Liouville's equation has been introduced by Wigner [3], [4] in the form

$$\begin{aligned}\frac{\partial W}{\partial t} &+ \frac{p}{m}\frac{\partial W}{\partial x} \\ &= \frac{1}{j\hbar} \\ &\cdot\left[V\left(x + \frac{j\hbar}{2}\frac{\partial}{\partial p}\right) - V\left(x - \frac{j\hbar}{2}\frac{\partial}{\partial p}\right)\right]W\end{aligned} \tag{2}$$

where $\hbar$ is a constant, $\hbar = h/(2\pi)$, and $h = 6.6 \times 10^{-34}$[J/s] is the Planck's constant. The momentum operator is denoted by $\hat{p}$, and its form is $\hat{p} = -j\hbar\nabla$ (for the 1-D case $\hat{p} = -j\hbar(\partial/\partial x)$) in the space of x ($\hat{x} = x$). Expanding $V[x \pm (j\hbar/2)(\partial/\partial p)]$ into a Taylor series around x we get

$$\frac{\partial W}{\partial t} + \frac{p}{m}\frac{\partial W}{\partial x} = V'(x)\frac{\partial W}{\partial p} - \frac{\hbar^2}{24}V^{(3)}(x)\frac{\partial^3 W}{\partial p^3} + \cdots \tag{3}$$

From (3), one may easily conclude that the classical Liouville equation (1) follows as a limit of its quantum mechanics extension. This limit appears if the potential is of the form $V(x) = V_0 + V_1 x + V_2 x^2$ (potential in the linear oscillator)

[1] Although Wigner was awarded the Nobel prize in 1963, his theory is not widely available in quantum mechanics textbooks.

or if the terms in (3) of order $\hbar^n$, for $n \geq 2$, are negligible. The expression

$$Q = \frac{1}{j\hbar}\left[V\left(x + \frac{j\hbar}{2}\frac{\partial}{\partial p}\right) - V\left(x - \frac{j\hbar}{2}\frac{\partial}{\partial p}\right)\right]W - V'(x)\frac{\partial W}{\partial p} = -\frac{\hbar^2}{24}V^{(3)}(x)\frac{\partial^3 W}{\partial p^3} + \cdots \tag{4}$$

may be understood as a quantum correction of the classical Liouville form [3], [4]. This is a significant property of the Wigner representation since it may be used to transform the solutions from the classical to the quantum forms [4] or to deal with problems with mixed (quantum and classical) variables.

In quantum mechanics, constant $\hbar$ appears as a key quantity in numerous relations. Here, we will mention (and use in the explanations that follow) only one, which is very familiar to electrical engineering specialists: An electromagnetic wave having wavenumber k may be treated as a particle with momentum $p = \hbar k$. When electrical and magnetic field components are normal to the propagation axis (TEM wave), we may also write $p = \hbar\sqrt{\epsilon\mu}\omega = \hbar_c\omega$, with $\hbar_c = \hbar\sqrt{\epsilon\mu}$. Constant $\hbar$ is present in the original (quantum mechanics) form of the Wigner distribution as well [3], [4]:

$$W(x, p, t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F\left(x + \frac{\hbar\xi}{2}, t\right) \cdot F^*\left(x - \frac{\hbar\xi}{2}, t\right)e^{-jp\xi}\, d\xi$$

or, for stationary problems

$$W(x, p) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \psi\left(x + \frac{\hbar\xi}{2}\right)\psi^*\left(x - \frac{\hbar\xi}{2}\right)e^{-jp\xi}\, d\xi \tag{5}$$

where $\psi(x, t) = F(x, t)\exp[j\Phi(t)/\hbar]$. Wavefunction ψ in the previous equations satisfies the Schroedinger equation

$$\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + j\hbar\frac{\partial\psi}{\partial t} = V(x)\psi \tag{6}$$

if $W(x, p, t)$ satisfies the Wigner quantum equation (2). It may be shown that the Wigner representation and the Schroedinger's one are equivalent, i.e., they uniquely follow from each other [3], [4].

Here, we will also indicate that *the uncertainty principle in the Wigner representation* states that

$$\sigma_x^2\sigma_p^2 \geq \frac{\hbar^2}{4}$$

where σ_x^2 and σ_p^2 are defined by

$$\sigma_q^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(q - \overline{q})^2 W(x, p)\, dx\, dp$$

$$\overline{q} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} qW(x, p)\, dx\, dp$$

when we substitute q with x and p, respectively. The product of the Wigner distribution standard deviations in directions of the x and p axis cannot be arbitrarily small. It is always greater or equal to $\hbar^2/4$, [4].

Any function

$$\psi(x) = A(x)e^{j\phi(x)/\hbar} = \psi_0^{[1/\hbar]}(x) \tag{7}$$

with $\psi_0(x) = A(x)e^{j\phi(x)}$ being $\hbar$-independent is the solution of the Schroedinger's (6) if

$$-[\phi'(x)]^2 = 2\text{ mV}(x). \tag{8}$$

and

$$\left|\frac{A''(x)}{A(x)}\hbar + j\left[\phi''(x) + 2\phi'(x)\frac{A'(x)}{A(x)}\right]\right|\hbar \ll |\phi'(x)|^2. \tag{9}$$

In light of (9), we mention again that $\hbar$ is of order 10^{-34}. The above forms are known as the pseudo classical approximation for stationary problems. Thus, for any function (7) satisfying (9), the Wigner distribution may be written in the form

$$W(x, p) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\psi_0^{[1/\hbar]}\left(x + \frac{\hbar\xi}{2}\right) \cdot \psi_0^{*[1/\hbar]}\left(x - \frac{\hbar\xi}{2}\right)e^{-jp\xi}\, d\xi. \tag{10}$$

The wavefunction defined by (7), along with (9) and (8) and with $A(x) = A$, is a form of the Schroedinger's equation solution, which was proposed by Wentzel [14]. It is efficiently used in quantum mechanics problems, especially for transmission coefficient calculations. Formally, the same form as (7), with $A(x) = A$, is used as a wavefunction in the famous Feynman's theory of path integrals [15] $\psi[x(t)] = A\exp(j(1/\hbar)S[x(t)])$ (where $S[x(t)] = \int_{t_1}^{t_2} L(dx/dt, x, t)\, dt$ is $\hbar$-independent, and $L = m(dx/dt)^2/2 - V(x, t)$ is the Lagrange's operator). The wavefunction in Bloch's theorem may be also written in form (7). We may conclude that (7), although not general, is a very common form of the wavefunction in quantum mechanics.

III. On the Signal Analysis Form of the Wigner Distribution

In signal analysis, the variables: frequency (ω) and time (t) are used, instead of p and x. In the time domain, the operators are given by $\hat{\omega} = -j(\partial/\partial t)$ and $\hat{t} = t$, [1]. The Wigner distribution of signal $s(t)$ with these coordinates is derived as

$$WD(t, \omega) = \int_{-\infty}^{\infty} s\left(t + \frac{\tau}{2}\right)s^*\left(t - \frac{\tau}{2}\right)e^{-j\omega\tau}\, d\tau. \tag{11}$$

The *formal mathematical correspondence* between the quantum mechanics definition (5) and the signal analysis definition (11) is obvious with $\psi(x) \to s(t)$, $x \to t$, $p \to \omega$, and $\hbar \to 1$. The presence of factor $1/2\pi$ is due to different forms of the Fourier transform commonly used in quantum mechanics and signal analysis (we intentionally did not want to modify any of them). This is a natural analogy that was used in the extension of quantum mechanics concepts and definitions to the signal analysis (see [1], [16]–[18]). Of course, in signal analysis, a signal need not to satisfy the Schroedinger equation (6). The signal is rather obtained as a result of some physical processes or theoretical analysis. Note also that quantum mechanics is an inherently probabilistic theory in contrast with signal analysis, which is a deterministic theory [1].

For a frequency-modulated signal $s(t) = A\exp[j\phi(t)]$, the Wigner distribution (11) assumes the form

$$\begin{aligned} WD(t, \omega) &= A^2 \int_{-\infty}^{\infty} e^{j[\phi(t+\tau/2)-\phi(t-\tau/2)]} e^{-j\omega\tau}\, d\tau \\ &= A^2 \int_{-\infty}^{\infty} e^{j[\phi(t+\tau/2)-\phi(t-\tau/2)]-j\phi'(t)\tau} \\ &\quad \cdot e^{j\phi'(t)\tau} e^{-j\omega\tau}\, d\tau. \end{aligned}$$

Factor $A^2 \int_{-\infty}^{\infty} e^{j\phi'(t)\tau} e^{-j\omega\tau}\, d\tau$ produces the ideal distribution concentration $2\pi A^2 \delta[\omega - \phi'(t)]$, whereas the term (whose phase is of the form that formally corresponds to the quantum correction factor (4))

$$\begin{aligned} Q &= j\left[\phi\left(t+\frac{\tau}{2}\right) - \phi\left(t-\frac{\tau}{2}\right)\right] - j\phi'(t)\tau \\ &= j\, \tfrac{1}{24}\, \phi^{(3)}(t)\tau^3 + \cdots \end{aligned} \tag{12}$$

produces the spread of distribution around the instantaneous frequency. Factor Q is equal to zero if instantaneous frequency $\phi'(t)$ is a linear function, i.e., if $\phi^{(n)}(t) \equiv 0$, for $n \geq 3$. In quantum mechanics, the quantum correction term Q was equal to zero for the potential function such that the terms with $\hbar^{n-1}$, $V^{(n)}(x)$, $n \geq 3$ are negligible. This is in agreement with (8), where linear function $\phi'(t)$ corresponds to quadratic function $V(x)$.

IV. "Pseudo Quantum" Signal Representation

We now pose the question: Is it possible to use a form of the Wigner distribution in the signal analysis other than (11)? In particular, we look after a form that would keep a constant corresponding to $\hbar$ in (5). This would be of great help in the signal analysis, especially since here, we are not restricted to the physical (real world) value of this constant. Thus, we would have an opportunity to choose its most suitable value. Now, we will present a reflection that led to that form of the Wigner distribution.

We have already mentioned that there is a complete *formal mathematical* correspondence between quantum mechanics and signal processing forms of the Wigner distribution. We have also shown how to get the signal analysis form from the quantum mechanics one. Now, we will consider the opposite direction, i.e., generation of a quantum mechanics form of the Wigner distribution from the signal analysis one. Of course, the correspondence is again only mathematical formality since the created "wave function" generally may not correspond to a true wave function nor the real world value of constant $\hbar$ will be acceptable for the analysis of signals. Let us, for the sake of argument, transcend the real world and enter the realm of a thought experiment. Assume that there are fictitious "spaces" in which $\hbar$ may assume some other constant values and not just the conventional one. This fictitious constants will be denoted by $\hbar_f$. Forms associated with this new constant $\hbar_f$ will be, in the sequel, referred to as "*pseudo quantum forms*." Having in mind this freedom, we may reinterpret the above signal processing definitions in the following way: On the basis of the *signal* given in the signal analysis, we mathematically generate the "*pseudo wave function*" with the corresponding "*pseudo particle*" having the "*pseudo-momentum*" $\wp = \hbar_f \omega$. Thus, signal analysis form of the Wigner distribution (11) may be treated as a special case of the "pseudo quantum" form of (5) with $\hbar_f = 1$ (in the "space" where $\wp \equiv \omega$). Now, we may pose the question: Why are we restricted to $\hbar_f = 1$, or is it possible to obtain any improvement in the signal analysis using some other values for $\hbar_f$?

It is obvious from the quantum mechanics forms that the uncertainty (which is now of order $\hbar_f^2$) may be decreased by using smaller values of $\hbar_f$. If we are able, for a given signal, to form a "pseudo wave function" having different "pseudo momentums" in different *fictitious* "spaces" (with different constants $\hbar_f$), then we can always go to a "space" with a small uncertainty and analyze the signal in that "space" (in its $(\wp, t)$ plane). As a measure of the representation quality, we may also consider the distribution concentration along the instantaneous frequency (group delay). This is especially interesting for a very important class of asymptotic signals when the uncertainty is always large, even for distributions that are very highly concentrated along the instantaneous frequency (group delay) [20]. For example, if a signal is linear frequency modulated, then the Wigner distribution (11) in the "space" with $\hbar_f = 1$ produces ideal concentration at its instantaneous frequency (12). Therefore, in this case, there is no need to go to any "space" with smaller $\hbar_f$ i.e., smaller uncertainty. However, if the signal is not linear frequency modulated, we should go to "space" with smaller $\hbar_f$ in order to improve the distribution concentration. How far we go with the decreasing of the $\hbar_f$ depends on how significantly the nonlinearities are exhibited in the frequency-modulated signal or, globally, how large the distribution uncertainty is. In this way, *by varying the value of $\hbar_f$, we are in aposition to influence the concentration (or uncertainty) of the "pseudo quantum" signal representation while, as it will be shown, keeping the most important properties of the time-frequency representation invariant*.

Here, we will present a distribution exhibiting the above properties. The idea for this distribution is based on the quantum forms given by (7) and (10). Transformation of a signal into the "pseudo wave function" is done according to (7) with $\psi_0(x) \leftrightarrow s(t)$, $x \leftrightarrow t$, $p \leftrightarrow \wp$, $\hbar \leftrightarrow \hbar_f$. Although the described transformation is only *one of the possibilities*, we preferred it because of its

1) quite general form with respect to signal functions,
2) simplicity,
3) presentation efficiency (Section V),
4) convenience in the numerical realization (Section VI).

The other quantum mechanics forms may produce other transformation schemes that will, hopefully, be worth future investigation for signal processing purposes (for example, using the wavefunction of the quantum linear oscillator as a transformation scheme, we would exactly get the L-Wigner distribution). According to the above considerations, (7) and (10) assume the signal processing shape

$$\mathrm{SD}(t, \wp) = \int_{-\infty}^{\infty} s^{[L]}\left(t + \frac{\tau}{2L}\right) s^{*[L]}\left(t - \frac{\tau}{2L}\right) e^{-j\wp\tau}\, d\tau \tag{13}$$

with $\psi_o(x)$ being replaced by signal $s(t)$ and $1/\hbar_f$ by L. The operators in the time domain are given by $\hat{\wp} = -(j/L)(\partial/\partial t)$ and $\hat{t} = t$.

For frequency-modulated signal $s(t) = A\exp[j\phi(t)]$, the term described by (12), which causes the distribution spread, is of the form

$$\begin{aligned} Q &= jL\Big[\phi\Big(t+\frac{\tau}{2L}\Big) - \phi\Big(t-\frac{\tau}{2L}\Big)\Big] - j\phi'(t)\tau \\ &= j\frac{1}{24L^2}\phi^{(3)}(t)\tau^3 + \cdots. \end{aligned} \tag{14}$$

For values $L > 1$, this factor is significantly reduced with respect to (12), meaning significant improvement of the distribution concentration along the instantaneous frequency.[2] The uncertainty limit of distribution (13) is also of order $1/L^2$. The definitions and properties of distribution (13), that will prove the above statements, are studied in the next section.

V. Definition and Properties of the SD

A. Definition

The *windowed* form of the *S*caled variant of the L-Wigner *D*istribution (*SD*) of signal $s(t)$ is defined by

$$\mathrm{SD}(t,\wp) = \int_{-\infty}^{\infty} w_L(\tau)s^{[L]}\Big(t+\frac{\tau}{2L}\Big)s^{*[L]}\Big(t-\frac{\tau}{2L}\Big)e^{-j\wp\tau}d\tau \tag{15}$$

where $s^{[L]}(t)$ is the modification of $s(t) = A(t)\exp[j\phi(t)]$ obtained by multiplying the phase function by L while keeping the amplitude unchanged

$$s^{[L]}(t) = A(t)e^{jL\phi(t)}. \tag{16}$$

Note that the word "windowed" will be used in front of the SD to indicate presence of window $w_L(\tau)$.

B. Properties

1°: The SD is always real.

2°: The distribution defined by (15) satisfies the time marginal and unbiased energy condition for any L:

$$\begin{aligned} \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{SD}(t,\wp)\,d\wp &= A^2(t) \\ &= |s(t)|^2 \\ \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{SD}(t,\wp)\,d\wp\,dt &= \int_{-\infty}^{\infty}A^2(t)\,dt \\ &= E_s \end{aligned} \tag{17}$$

where $w_L(0) = 1$ is assumed.

3°: For asymptotic signals (signals whose phase variations are much faster than its amplitude variations [20]), the frequency marginal

$$\int_{-\infty}^{\infty}\mathrm{SD}(t,\wp)\,dt = |\mathbf{S}(\wp)|^2$$

is satisfied, as well. Fourier transform of $s(t)$ is denoted by $\mathbf{S}(\wp)$. Substituting τ/L by τ in (13), we have

$$\begin{aligned} \mathrm{SD}(t,\wp) &= L\int_{-\infty}^{\infty}A\Big(t+\frac{\tau}{2}\Big)A\Big(t-\frac{\tau}{2}\Big) \\ &\quad\cdot e^{jL[\phi(t+\tau/2)-\phi(t-\tau/2)-\wp\tau]}\,d\tau \\ &= \frac{L}{2\pi}\int_{-\infty}^{\infty}\mathbf{S}_L\Big(L\wp+\frac{\theta}{2}\Big)\mathbf{S}_L^*\Big(L\wp-\frac{\theta}{2}\Big)e^{j\theta t}\,d\theta \end{aligned} \tag{18}$$

where $\mathbf{S}_L(\wp) = FT\{A(t)\exp(jL\phi(t))\}$. According to the stationary phase method [20] for asymptotic signals $s(t)$, it holds that

$$\mathbf{S}_L(L\wp) \cong A(t_0)e^{jL[\phi(t_0)-\wp t_0]}\sqrt{\frac{2\pi}{L|\phi'(t_0)|}}\,e^{\pm j\pi/4}$$

where t_0 is the solution of $\phi'(t_0) = \wp$. Thus, we get $L|\mathbf{S}_L(L\wp)|^2 \cong |\mathbf{S}(\wp)|^2$, and the proof directly follows [22]. Note that for asymptotic signals

$$\mathbf{S}_L(L\wp) = \frac{1}{\sqrt{L}}\mathbf{S}^{[L]}(\wp)e^{\mp j\pi(L-1)/4}.$$

For these signals, according to (18), we have

$$\mathrm{SD}(t,\wp) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathbf{S}^{[L]}\Big(\wp+\frac{\theta}{2}\Big)\mathbf{S}^{[L]*}\Big(\wp-\frac{\theta}{2}\Big)e^{j\theta t}\,d\theta. \tag{19}$$

This relation is dual to (13). It means that all properties that hold in the time (t) domain will be valid, for asymptotic signals, in the pseudo momentum $(\wp)$ domain.

4°: The time moments property

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}t^n\mathrm{SD}(t,\wp)\,d\wp\,dt = \int_{-\infty}^{\infty}t^n|s(t)|^2\,dt$$

is satisfied by (13) for any L since $s^{[L]}(t)s^{[L]*}(t) = |s(t)|^2$. The frequency moments property is satisfied for asymptotic signals.

5°: The mean conditional value of the SD, at a particular instant t, defined by

$$\langle\wp\rangle_t = \frac{\frac{1}{2\pi}\int_{-\infty}^{\infty}\wp\mathrm{SD}(t,\wp)\,d\wp}{\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{SD}(t,\wp)\,d\wp}$$

and is invariant with respect to L. It is equal to the signal's instantaneous frequency

$$\langle\wp\rangle_t = \langle\omega\rangle_t = \phi'(t).$$

Proof: Observing that

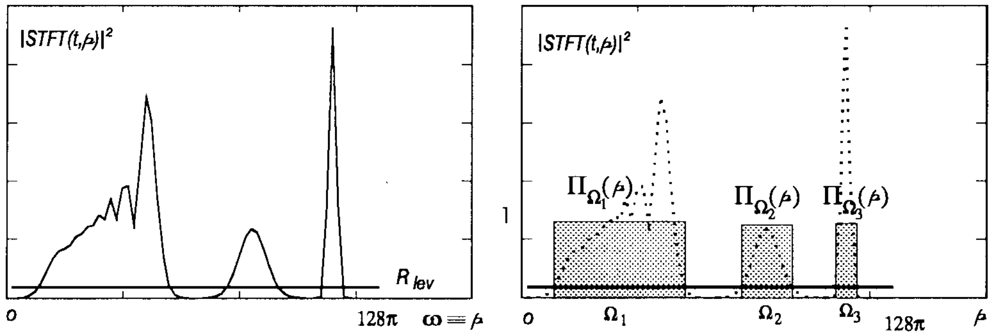
$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{SD}(t,\wp)\,d\wp = A^2(t)$$

and

$$\begin{aligned} &\frac{j}{2\pi}\int_{-\infty}^{\infty}\wp\mathrm{SD}(t,\wp)\,d\wp \\ &\quad= \frac{\partial}{\partial\tau}\Big[s^{[L]}\Big(t+\frac{\tau}{2L}\Big)s^{*[L]}\Big(t-\frac{\tau}{2L}\Big)\Big]_{|\tau=0} \end{aligned}$$

the proof directly follows. Note that for $L = 1$, $\wp \equiv \omega$. For asymptotic signals dual relation holds for the group delay.

[2] The true (real-world) value of $\hbar$, along with its quantum mechanics form, would require a signal's power of order $1/\hbar_c \sim 10^{42}$. It is absolutely unacceptable in the signal analysis. The values that are used in our papers ranges from $L = 1/\hbar_f = 1$ up to $L = 1/\hbar_f = 8$.



(a) (b)

Fig. 1. Illustration of the regions of support. (a) Spectrogram at a given instant t. (b) Regions of support with the reference level.

6^o Theorem: For any signal $s(t) = A(t)\,\exp{[j\phi(t)]}$ having finite derivatives of the phase function $\phi(t)$ and continuous amplitude $A(t)$, the SD for $L \to \infty$ is completely concentrated along the instantaneous frequency

$$\lim_{L\to\infty} \mathrm{SD}(t,\,\wp) = A^2(t)W[\wp - \phi'(t)] \tag{20}$$

where $w(\tau)$ is a finite duration window $W(\wp) = FT\{w(\tau)\}$.

The proof may be found in [22].

7^o: If a signal $\tilde{s}(t)$ is a time-shifted version of $s(t)\,[\tilde{s}(t) = s(t - t_o)]$, then $\mathrm{SD}_{\tilde{s}}(t, \wp) = \mathrm{SD}_s(t - t_o, \wp)$ for any L.

8^o: The SD of a modulated signal $s(t)\,\exp{(j\omega_o t)}$ is $\mathrm{SD}_s(t, \wp - \omega_o)$.

9^o: The SD is limited to the same time interval as the signal itself: If $s(t) = 0$ for $|t| > T$, then $\mathrm{SD}(t, \wp) = 0$ for $|t| > T$.

10^o: The SD is limited in the $\wp$ direction for asymptotic signals and any L to the same interval as $\mathbf{S}(\wp) = FT\{s(t)\}$, i.e., if $\mathbf{S}(\wp) = 0$ for $|\wp| > W$, then $\mathrm{SD}(t, \wp) = 0$ for $|\wp| > W$. The proof follows from (19).

11^o: If we have a product of two signals $s(t)h(t)$, then its SD is equal to the convolution in frequency of the SD's of each signal separately. The SD of the product of Fourier transforms of two asymptotic signals is equal to the time-domain convolution of the distributions of each signal.

12^o: If a signal $s(t)$ is multiplied by the chirp signal $\exp{(jct^2/2)}$, then its SD is $\mathrm{SD}_s(t, \wp - ct)$ for any L. If $s(t) = A(t)\,\exp{[j\phi(t)]}$ is multiplied by an arbitrary frequency-modulated signal $\exp{[j\gamma(t)]}$ satisfying the conditions of Theorem in 6^o, then for large L, we get that $\mathrm{SD}(t, \wp) = A^2(t)W[\wp - \phi'(t) - \gamma'(t)]$. In a dual form, the same is valid for asymptotic signals.

13^o: A 2-D real function $P(t, \wp)$ is the SD distribution of a signal if

$$\frac{\partial^2 \ln{[\rho(t_1,\, t_2)]}}{\partial t_1 \partial t_2} = 0 \tag{21}$$

where

$$\rho(t_1,\, t_2) = \frac{1}{2\pi}\int_{-\infty}^{\infty} P\left(\frac{t_1 + t_2}{2},\, \wp\right) e^{j\wp L(t_1 - t_2)}\, d\wp.$$

Proof: If (21) is satisfied, then $\ln{[\rho(t_1,\, t_2)]} = \varphi_1(t_1) + \varphi_2(t_2)$, where $\varphi_1(t_1)$ and $\varphi_2(t_2)$ are arbitrary functions of t_1 and t_2. It follows that $\rho(t_1,\, t_2) = e^{\varphi_1(t_1)}e^{\varphi_2(t_2)} = f_1(t_1)f_2(t_2)$. Since, for any function $f(t)$, there exists a function $s(t)$ such that $f(t) = s^{[L]}(t)$, we may write $\rho(t_1,\, t_2) = s_1^{[L]}(t_1)s_2^{[L]}(t_2)$. From (21), with $t_1 = t + \tau/2L$ and $t_2 = t - \tau/2L$, we get

$$s_1^{[L]}\left(t + \frac{\tau}{2L}\right)s_2^{[L]}\left(t - \frac{\tau}{2L}\right) = \frac{1}{2\pi}\int_{-\infty}^{\infty} P(t,\,\wp)e^{j\wp\tau}\, d\wp.$$

Since $P(t, \wp)$ is a real function, it follows that $s_1^{[L]}(t) = As_2^{*[L]}(t) = \sqrt{A}s^{[L]}(t)$, where A is a real constant. Thus, for $P(t, \wp)$ satisfying (21), there exists function $s^{[L]}(t)$ such that $s^{[L]}[t + (\tau/2L)]s^{*[L]}[t - (\tau/2L)]$ and $P(t, \wp)$ are the Fourier transform pair. Q.E.D.

14^o Generalization: Form (15) may be applied to any distribution from the Cohen class, defining a class of distributions

$$\begin{aligned} SC(t,\,\wp) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} & s^{[L]}\left(u + \frac{\tau}{2L}\right) \\ & \cdot s^{*[L]}\left(u - \frac{\tau}{2L}\right)c_L(\theta,\,\tau) \\ & \cdot e^{-j\theta t - j\wp\tau + j\theta u}\, d\tau\, d\theta\, du. \end{aligned} \tag{22}$$

Properties of $SC(t, \wp)$ are studied in detail in [23].

VI. On the Realization

Direct realization is based on the straightforward application of the SD definition (15), (16). Signal $s(t)$ should be modified into $s^{[L]}(t)$ and oversampled L times, whereas the number of samples used for calculation is kept unchanged. Regarding the last assumption, this method is not computationally much more demanding than the realization of ordinary ($L = 1$) Wigner distribution. In the case of multicomponent signals, this method will produce signal power concentrated at the resulting instantaneous frequency, according to the Theorem in 6^o. Some examples with the direct method of realization are presented in [22]. Here, we will present a method for the SD realization that will be efficient in the case of multicomponent signals. This method will also provide that neither oversampling (with respect to the Nyquist rate) nor an analytic signal application is needed for the realization.

Consider a *multicomponent signal*

$$s(t) = \sum_{i=1}^{P} s_i(t).$$

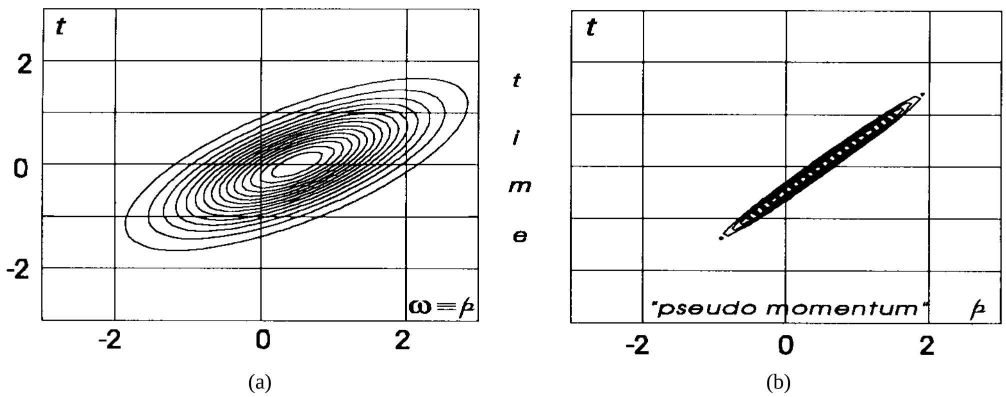


Fig. 2. Time-frequency (pseudo-quantum) representation of a Gaussian chirp signal: (a) Wigner distribution and (b) the SD with $L = 8$. ($A = a = b = 2c = 1$).

Our *aim* is to obtain a distribution such that it is theoretically equal to the sum of the SD's of each component separately, i.e.

$$\mathrm{SD}_s(t, \wp) = \sum_{i=1}^{P} \mathrm{SD}_{s_i}(t, \wp). \tag{23}$$

Note that the marginal conditions in this case will be $\sum_{i=1}^{P} |s_i(t)|^2$ and $\sum_{i=1}^{P} |\mathbf{S}_i(\wp)|^2$. Let us start from the short-time Fourier transform of $s(t)$ in the space with $L = 1$ (and $\wp \equiv \omega$):

$$\begin{aligned}\mathrm{STFT}(t, \wp) &= \int_{-\infty}^{\infty} w(\tau) s(t+\tau)\, e^{-j\wp\tau}\, d\tau \\ &= \int_{-\infty}^{\infty} w(\tau) A(t+\tau)\, e^{j\phi(t+\tau)}\, e^{-j\wp\tau}\, d\tau. \end{aligned} \tag{24}$$

As it is known, this transform does not have crossterms between signal components that are separated in the time-frequency plane. In order to produce a higher order SD, we will need an amplitude normalized version of $\mathrm{STFT}(t, \wp)$, which will be denoted by $\mathrm{STFT}_n(t, \wp)$ and defined as

$$\mathrm{STFT}_n(t, \wp) = \int_{-\infty}^{\infty} w(\tau)\, e^{j\phi(t+\tau)}\, e^{-j\wp\tau}\, d\tau.$$

If amplitude $A(t)$ is slow varying, we may easily get $\mathrm{STFT}_n(t, \wp)$ from $\mathrm{STFT}(t, \wp)$ as

$$\mathrm{STFT}_n(t, \wp) = \mathrm{STFT}(t, \wp) \sqrt{\frac{E_w}{E_s(t)}} \tag{25}$$

where $E_s(t) = (1/2\pi) \int_{-\infty}^{\infty} |\mathrm{STFT}(t, \wp)|^2\, d\wp$, and E_w is the energy of window $w(\tau)$. In the derivation of (25), the Parseval's theorem

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} |\mathrm{STFT}(t, \wp)|^2\, d\wp = \int_{-\infty}^{\infty} |w(\tau) A(t+\tau)|^2\, d\tau$$

with $w(\tau)A(t+\tau) \cong w(\tau)A(t)$ is used.[3]

[3] Slow-varying amplitude $A(t)$ means that $w(\tau)A(t+\tau) \approx w(\tau)A(t)$. This condition may be written in a less restrictive form. Assume, for example, a Hanning window $w(\tau)$ and $A(t+\tau) = A(t) + A'(t)\tau + A''(t)\tau^2/2$. The scaling factor in (25) remains the same if $A^2(t) \gg [A'^2(t) + A(t)A''(t)]/6.17 + A''^2(t)/120$, i.e., if $A(t)$, $A'(t)$, $A''(t)$ are of the same order. In the examples, we will see that the results will not be significantly degraded even if this condition is not satisfied.

If the signal is multicomponent, with slow-varying amplitudes of each component, and the components are separated along the frequency axis for any t (i.e., they lie along $\wp$ inside regions Ω_i which do not overlap), then according to (25)

$$\mathrm{STFT}_n(t, \wp) = \sum_{i=1}^{P} \mathrm{STFT}(t, \wp) \sqrt{\frac{E_w}{E_{s_i}(t)}}\, \Pi_{\Omega_i}(\wp) \tag{26}$$

where $E_{s_i}(t) = (1/2\pi) \int_{-\infty}^{\infty} |\mathrm{STFT}(t, \wp)|^2 \Pi_{\Omega_i}(\wp)\, d\wp$, and $\Pi_{\Omega_i}(\wp)$ is equal to unity for $\wp$ inside Ω_i and zero outside (regions Ω_i are numerically determined as the compact regions where $|\mathrm{STFT}(t, \wp)|^2$ is greater than an assumed reference level R_{lev}; see Fig. 1). Details may be found in [21], [23], and [24]. In order to realize the SD, let us define an intermediate distribution as

$$\begin{aligned} S_I(t, \wp) = \int_{-\infty}^{\infty} & w^2\Big(\frac{\tau}{2}\Big) A\Big(t + \frac{\tau}{2}\Big) e^{j\phi(t+\tau/2)} \\ & \cdot e^{-j\phi(t-\tau/2)} e^{-j\wp\tau}\, d\tau \end{aligned}$$

with $w(\tau) = w(-\tau)$. Knowing $\mathrm{STFT}(t, \wp)$ and $\mathrm{STFT}_n(t, \wp)$, we may easily realize a crossterms-free version of $S_I(t, \wp)$ using the S method:

$$\begin{aligned} S_I(t, \wp) = \int_{-\infty}^{\infty} & P(\theta) \mathrm{STFT}(t, \wp + \theta) \\ & \cdot \mathrm{STFT}_n^*(t, \wp - \theta)\, \frac{d\theta}{\pi} \end{aligned} \tag{27}$$

where $P(\theta)$ is a frequency domain window function, which has to be wide enough to ensure the integration over autoterms and narrow enough to avoid crossterms (see [7], [8], [12], [13], [21], [23], and [24] for details). For a given $\wp$, inside Ω_i, the optimal window $P(\theta)$ width is determined by the width of product $\Pi_{\Omega_i}(\wp + \theta)\Pi_{\Omega_i}(\wp - \theta)$ along θ. Note that in general, this product is $\wp$ and t dependent, and so is the window $P(\theta)$ width [21], [23], [24]. After we get crossterms-free $S_I(t, \wp)$, then we may get the SD (15) for $L = 2$:

$$\begin{aligned} \mathrm{SD}(t, \wp) = \int_{-\infty}^{\infty} & w^4\Big(\frac{\tau}{4}\Big) A\Big(t + \frac{\tau}{4}\Big) A\Big(t - \frac{\tau}{4}\Big) \\ & \cdot e^{j2\phi(t+\tau/4)} e^{-j2\phi(t-\tau/4)} e^{-j\wp\tau}\, d\tau \end{aligned}$$

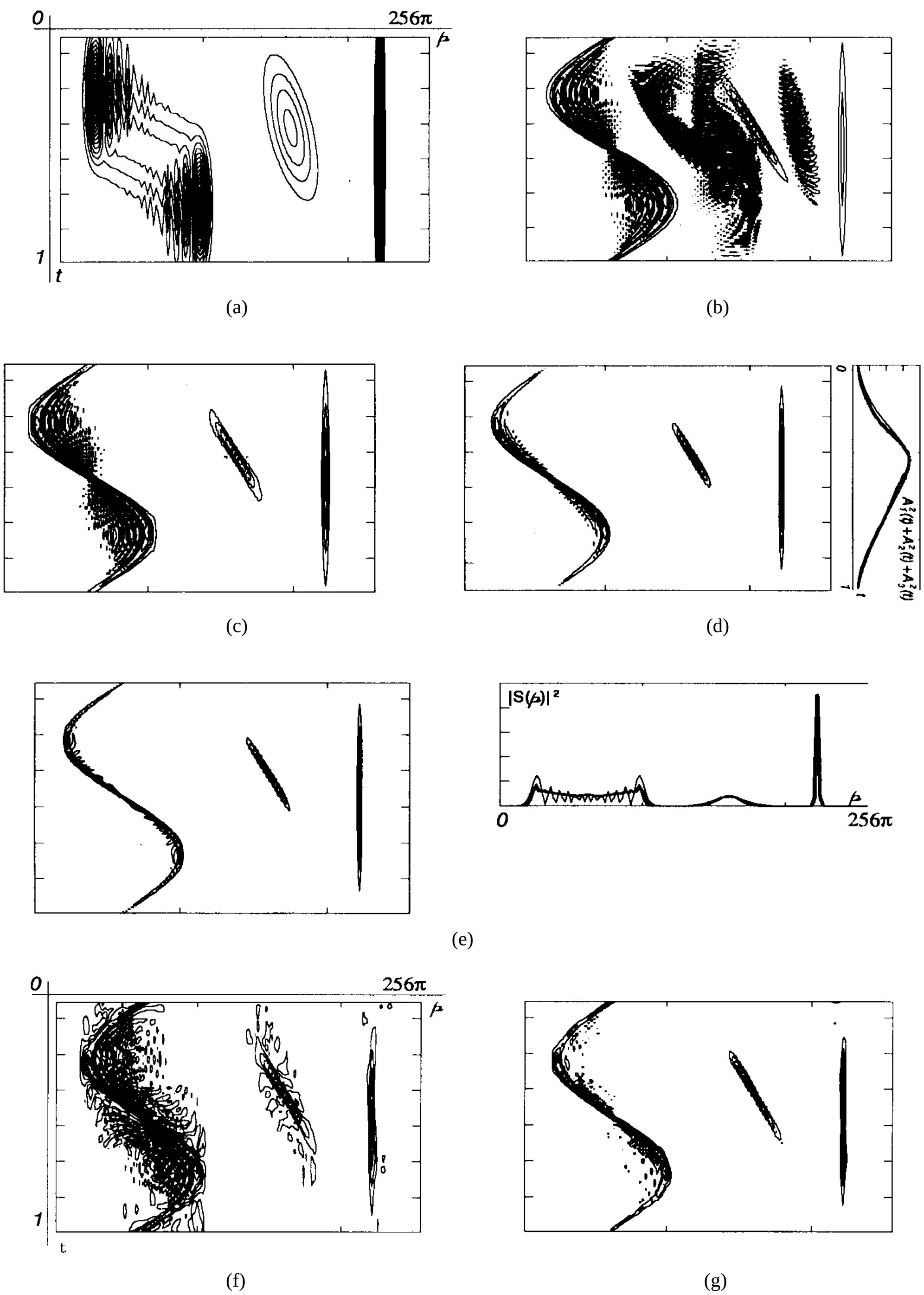


Fig. 3. Time-frequency (pseudo-quantum) representation of a multicomponent signal: (a) spectrogram, (b) Wigner distribution of an analytic signal, (c) S-method, (d) the SD with $L = 2$ along with marginal properties, (e) the SD with $L = 4$, (f) the s-method of noisy signal, and (g) the SD with $L = 2$ of noisy signal.

convolving two $S_I(t, \wp)$

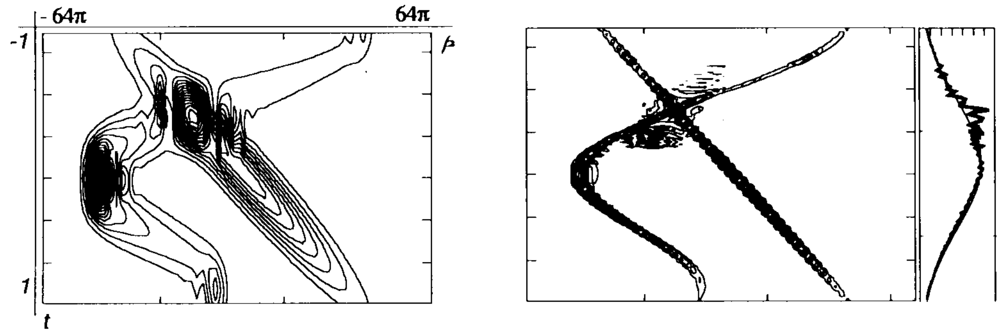
$$\mathrm{SD}(t, \wp) = \frac{1}{\pi} \int_{-\infty}^{\infty} P(\theta) S_I(t, \wp + \theta) S_I^*(t, \wp - \theta)\, d\theta \quad (28)$$

where, again, $P(\theta)$ eliminates (reduces) crossterms, whereas the autoterms are the same as in the original SD of order 2. This procedure may be continued up to any order of the SD. Namely, convolving $\mathrm{SD}(t, \wp)$ given by (28) and its normalized version $\mathrm{SD}_n(t, \wp)$, we get $\mathrm{SD}(t, \wp)$ with $L = 4$, and so on. Efficiency of the proposed realization will be demonstrated in the next section. Further details on the numerical realization may be also found in [21], as well as in [23] (along with a very simple MATLAB program for the SD realization).

VII. Examples

Example 1: The SD of a *Gaussian chirp signal*

$$s(t) = Ae^{-at^2/2} e^{jbt^2/2 + jct}$$



(a) (b)

Fig. 4. Time-frequency (pseudo-quantum) representation of a multicomponent signal whose components intersect. (a) Spectrogram and (b) the SD with $L = 2$ along with the time marginal property.

according to (13), is given by

$$\begin{aligned}\mathrm{SD}(t, \wp) &= \int_{-\infty}^{\infty} s^{[L]}\Big(t+\frac{\tau}{2L}\Big)s^{*[L]}\Big(t-\frac{\tau}{2L}\Big)e^{-j\wp\tau}\,d\tau\\ &= A^2e^{-at^2}\sqrt{\frac{4\pi}{a}}\,Le^{-(\wp-bt-c)^2/(a/L^2)}.\end{aligned}$$

For $a/L^2 \to 0$, we get $\mathrm{SD}(t, \wp) = 2\pi A^2 e^{-at^2}\delta(\wp - bt - c)$, which is just an ideally concentrated distribution along the instantaneous frequency; see Fig. 2. For a large a, if L^2 is large enough so that $a/L^2 \to 0$, we get the distribution highly concentrated in a very small region around the point $(t, \wp) = (0, c)$.

Example 2: Consider a *multicomponent real signal*

$$\begin{aligned}s(t) = &e^{-4(t-0.5)^2}\cos[20\cos(2\pi t) + 60\pi t]\\ &+ 0.5e^{-4(t-0.5)^2}\cos(220\pi t)\\ &+ 0.707e^{-20(t-0.4)^2}\cos(48\pi t^2 + 120\pi t) + n(t)\end{aligned}$$

where $n(t)$ is a Gaussian white noise. Hanning window $w(\tau)$ of the unity width with $N = 256$ samples, as well as a rectangular window $P(\theta)$, are used. Regions Ω_i (and window $P(\theta)$ widths) are determined on the basis of a reference level equal to 4% of the maximal spectrogram value for each considered t; see Fig. 1. The spectrogram of $s(t)$ is shown in Fig. 3(a). Fig. 3(b) and (c) present the Wigner distribution, as well as the S-method (autoterms as in the Wigner distribution but without cross-terms [8], [12], [13]). In the case of the Wigner distribution, an analytic part of the signal is used. The crossterms-free SD with $L = 2$ realized according to the procedure described in Section VI is shown in Fig. 3(d). Here, we also presented the marginals obtained from the SD (thick line), as well as the theoretical ones, according to (23) (thin lines). Note that $w(\tau)$ does not influence the time marginal condition, whereas the frequency marginal is smoothed by the Fourier transform of the resulting window in the SD. Further concentration improvement may be achieved using the SD with $L = 4$; see Fig. 3(e). Note that the third signal component (very short chirp pulse) is far from satisfying the condition of constant amplitude within the window $w(\tau)$, which is required by (25), but nevertheless, the representation of this component is in complete agreement with the presented theory.

Example 3—Multicomponent Real Noisy Signal: The case with a high amount of noise $n(t)$ (SNR = 4[dB]) added to the signal from *Example 2* is presented in Fig. 3(f) and (g). SNR is the ratio of total signal and noise energy within the considered time and frequency interval. For each separate signal component, this ratio would be 3, −3, and −12 dB, respectively. The S-method of noisy signal is presented in Fig. 3(f), whereas Fig. 3(g) shows the second-order SD, realized according to the procedure described in Section VI.

Example 4: Finally, assume a *multicomponent signal whose components intersect*:

$$s(t) = e^{-t^2}\big[e^{j20\pi t^2} + e^{-j32\,\sin(\pi t) - j3\pi t(5+4t)}\big].$$

In the realization of time-frequency presentation of this signal, the Hanning unity width window is used. The number of samples was $N = 64$. Rectangular window $P(\theta)$ with signal dependent width, in (27), as well as $P(\theta) = \pi\delta(\theta)$ in (28), are used. The spectrogram is shown in Fig. 4(a), whereas Fig. 4(b) presents the SD with $L = 2$. It may be observed that the distribution presented in Fig. 4(b) behaves according to the presented theory everywhere except around the point of intersection, where crossterms appear. They occurred because the support regions Ω_1 and Ω_2 overlap in neighborhood of this point, where both components are considered to be a single component, with all consequences as in the case of Wigner distribution.

VIII. Conclusion

Pseudo quantum signal representation is presented. One possible scheme for the transformation of a signal into "pseudo wave function" is presented. On the basis of that analysis, the distribution for time-frequency analysis (SD) is introduced. This distribution may produce an ideal concentration at the instantaneous frequency in the case of nonlinear frequency-modulated signals (or, generally, may produce representation with arbitrary small uncertainty limit), satisfying the marginal properties.

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