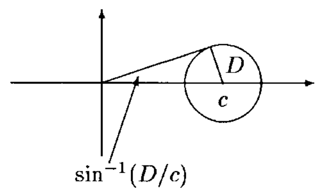
For (3) drop the argument f and write

$$\begin{aligned} D &= |T_d - A_0| \\ A_0 &= |A_0| e^{j\angle A_0} \\ T_d &= |T_d| e^{j\angle T_d} \\ &= c e^{j\angle T_d}. \end{aligned}$$

Then

$$\begin{aligned} D &= |c e^{j\angle T_d} - |A_0| e^{j\angle A_0}| \\ &= |c - |A_0| e^{j(\angle A_0 - \angle T_d)}| \end{aligned}$$

which says that the point $|A_0| e^{j(\angle A_0 - \angle T_d)}$ lies on the circle with center c, radius D:



Therefore

$$|\angle A_0 - \angle T_d| \le \sin^{-1} \frac{D}{c}.$$

■

Proof of Theorem 1: An outline of the proof is given here; details can be found in [2].

The alias component (AC) matrix for the M-channel filter bank is

$$H_{\mathrm{AC}}(f) = \frac{1}{M} \sum_{k=0}^{M-1} \begin{bmatrix} G_k(f) \\ G_k\left(f - \frac{1}{M}\right) \\ \vdots \\ G_k\left\{f - \frac{(M-1)}{M}\right\} \end{bmatrix} \times \left[F_k(f) \quad F_k\left(f - \frac{1}{M}\right) \quad \cdots \quad F_k\left\{f - \frac{(M-1)}{M}\right\} \right]$$

that is, the $n\ell$-element of $H_{AC}(f)$ is

$$h_{n\ell}(f) := \frac{1}{M} \sum_{k=0}^{M-1} G_k\left(f - \frac{n}{M}\right) F_k\left(f - \frac{\ell}{M}\right).$$

Also, let $P(f)$ denote the AC matrix for the ideal system $\mathbf{T}_d$, that is

$$P(f) = \begin{bmatrix} T_d(f) & 0 & \cdots & 0 \\ 0 & T_d\left(f - \frac{1}{M}\right) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & T_d\left\{f - \frac{(M-1)}{M}\right\} \end{bmatrix}.$$

Since the gain $G(\ell_2)$ of a linear periodically time-varying system equals the $\mathcal{L}_\infty$ norm of its AC matrix and since the AC matrix of $\mathbf{T}_d - \mathbf{T}$ equals $P(f) - H_{\mathrm{AC}}(f)$, we have that

$$J = \|P - H_{\mathrm{AC}}\|_\infty. \tag{4}$$

The elements on the first row of $H_{\mathrm{AC}}(f)$ are

$$\begin{aligned} h_{0\ell}(f) &= \frac{1}{M} \sum_{k=0}^{M-1} G_k(f) F_k\left(f - \frac{\ell}{M}\right) \\ &= A_\ell(f). \end{aligned}$$

Since the $\mathcal{L}_\infty$ norm of each block of a matrix is less than or equal to the $\mathcal{L}_\infty$ norm of that matrix, by considering the first row of $P(f) - H_{\mathrm{AC}}(f)$, we have from (4) that

$$\|[T_d - A_0 \quad -A_1 \quad \cdots \quad -A_{M-1}]\|_\infty \le J$$

that is

$$|T_d(f) - A_0(f)|^2 + \sum_{k=1}^{M-1} |A_k(f)|^2 \le J^2, \quad \forall f.$$

This is the same as saying

$$D(f)^2 + AD(f)^2 \le J^2, \quad \forall f.$$

Maximizing over f gives the inequality to be proved. ■

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An Architecture for the Realization of a System for Time-Frequency Signal Analysis

Srdjan Stanković and LJubiša Stanković

***Abstract*— An architecture of the system for time-frequency signal analysis is presented. This system is based on the S-method, whose special cases are two the most important distributions: the Spectrogram and the Wigner distribution. Systems with constant and signal-dependent window widths are presented.**

***Index Terms*— Spectrogram, time-frequency analysis, VLSI architecture, Wigner distribution.**

I. Introduction

Two of the most important and widely used distributions for time-frequency signal analysis are: the Spectrogram (SPEC) and the Wigner distribution (WD), along with its pseudo and smoothed forms [1], [2]. The S-method, recently defined in [3], and analyzed in details in [4]–[9], may produce the representation of a multicomponent signal

Manuscript received May 2, 1995; revised August 16, 1996. This paper was recommended by Associate Editor S. Kiaei.

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Publisher Item Identifier S 1057-7130(97)03647-1.

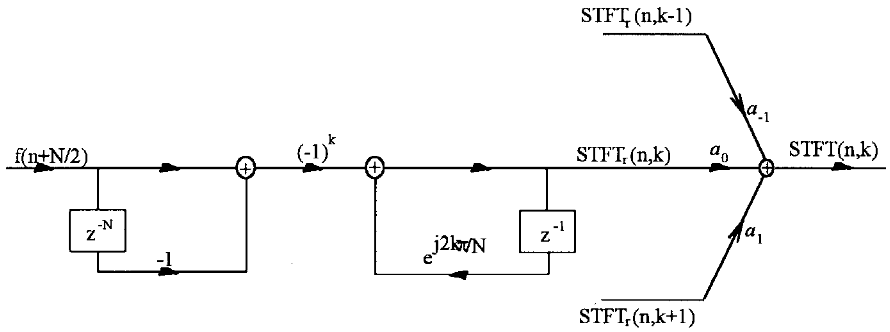

Fig. 1. System for the recursive STFT realization.

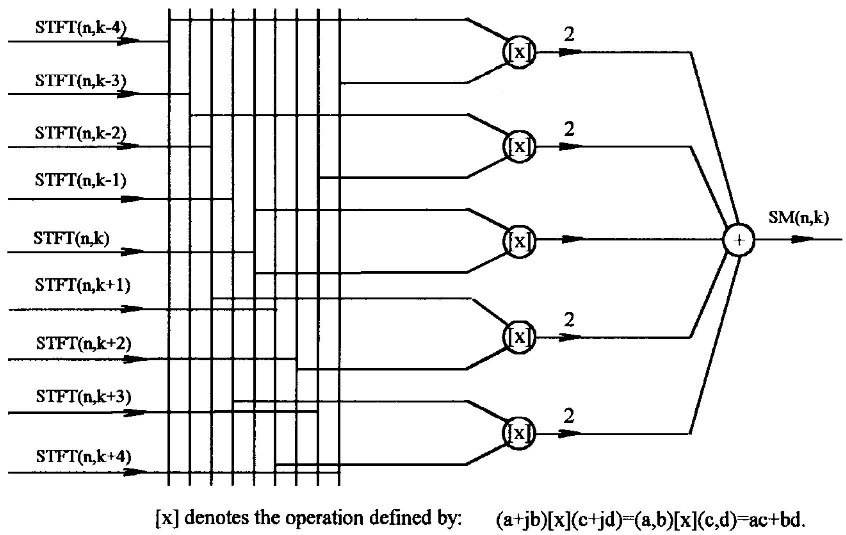

Fig. 2. Realization of the S-method with constant window $P(i)$ width, $2L_d + 1 = 9$.

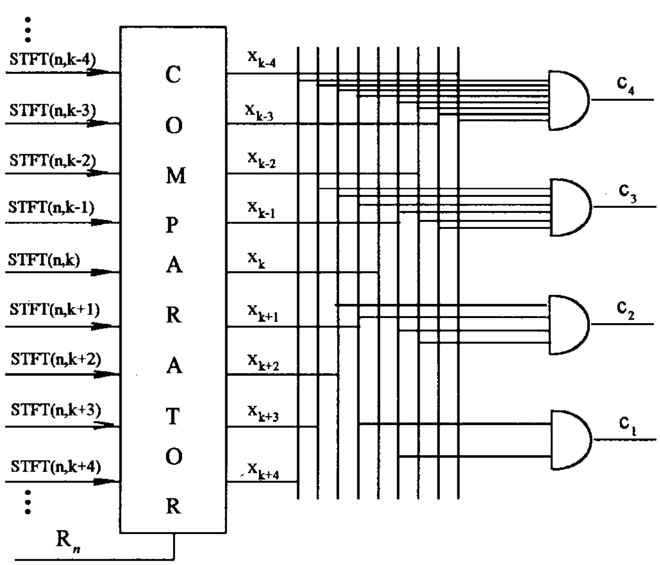

Fig. 3. Realization of control signals for the variable width of $P(i)$.

such that the distribution of each component is its WD [4]–[6], [8] but avoiding serious drawbacks of the WD: 1) Cross-terms are removed (or significantly reduced) [3]–[8]; 2) Signal over sampling, with respect to the sampling theorem, is not necessary [4], [6], [8]; and 3) Performances in the noisy environment are improved [9]. The S-method may be realized in a numerically very efficient way (more efficient than the WD realization itself) [3]. Two special (marginal) cases of the S-method, which readily follow, are just two the most frequently used distributions (the SPEC and the WD). In this brief, systems for hardware realizations of the S-method, with constant and variable window widths, are presented. It is shown that all appealing properties of the S-method may be further (significantly) improved using the variable width windows.

II. Review of the S-Method

The discrete form definitions of the Short-time Fourier transform (STFT), whose squared magnitude is called spectrogram, and the pseudo-Wigner distribution, are given by [1]–[4]

$$\mathrm{STFT}(n,\,k) = \sum_{i=-(N/2)+1}^{N/2} f(n+i)w(i)e^{-j(2\pi/N)ik} \tag{1}$$

$$\mathrm{WD}(n,\,k) = \sum_{i=-(N/2)+1}^{N/2} f(n+i)f^*(n-i) \cdot w(i)w(-i)e^{-j(2\pi/N)2ik}. \tag{2}$$

In the WD definition we omitted a factor of 2 in order to simplify the notation, as well as assumed a real window $w(i)$. Relation between (1) and (2) may be easily derived as

$$\mathrm{WD}(n,k) = \frac{1}{N}\sum_{i=-(N/2)+1}^{N/2} \alpha(i)\,\mathrm{STFT}(n,\,k+i) \cdot \mathrm{STFT}^*(n,k-i) \tag{3}$$

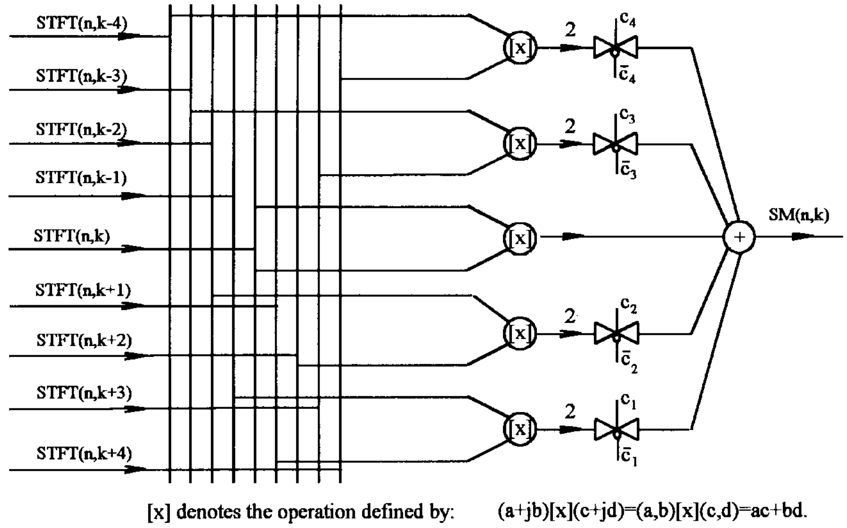

Fig. 4. Realization of the S-method with variable window $P(i)$ width and $2L_{\rm dmax} = 9$.

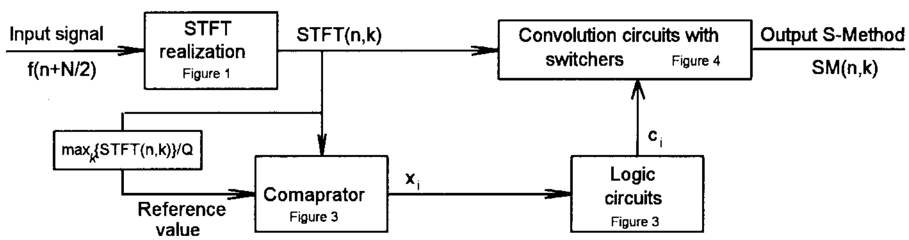

Fig. 5. Block diagram of a complete system for time-frequency signal analysis.

where $\alpha(i) = 1$, for all i except $|i| = N/2$, when $\alpha(\pm N/2) = \frac{1}{2}$. Relation (3) was used as a basis for the derivation of the S-method, whose discrete form is defined by

$$\mathrm{SM}(n,k) = \sum_{i=-L_d}^{L_d} P(i)\,\mathrm{STFT}(n,k+i)\,\mathrm{STFT}^*(n,k-i). \quad (4)$$

The S-method, for a real and symmetric window $P(i) = P^*(-i)$, may also be written in the form

$$\mathrm{SM}(n,\,k) = \mathrm{SPEC}(n,\,k) + 2\sum_{i=1}^{L_d} \cdot \mathrm{Re}\,\{P(i)\,\mathrm{STFT}(n,\,k+i)\,\mathrm{STFT}^*(n,\,k-i)\}. \quad (5)$$

Through a suitable selection of the window $P(i)$ it is possible to get the autoterms of multicomponent signals such that they remain unchanged with respect to the WD, while the entire elimination (or reduction) of cross-terms is reached (more details on the window $P(i)$may be found in [3]–[9], as well as in Section IV). Let us observe that: 1) for $P(i) = \delta(i)$ we get the SPEC; and 2) for $P(i) = 1/N$ the WD follows [in this case $\alpha(i)$ should be included].

III. Architecture for Signal Independent Realization

Here, we will present a system for the S-method realization using signal independent windows. First, a signal has to be transformed into the STFT. Assuming a rectangular window $w(i)$, this may be done in a recursive manner [3], [4], [6], [10], [11], [16], [17] according to

$$\mathrm{STFT}_r(n,\,k) = \left[f\left(n+\frac{N}{2}\right) - f\left(n-\frac{N}{2}\right)\right](-1)^k + \mathrm{STFT}_r(n-1,\,k)e^{j2\pi k/N}. \quad (6)$$

Architecture of the system, for a given k, is given in Fig. 1. Complete system contains N of these blocks, with $k = 0, 1, 2, \cdots N-1$. The coefficients a_{-1}, a_0, a_1 are, for example, (0, 1, 0), (0.25, 0.5, 0.25), or (0.23, 0.54, 0.23) for the rectangular, Hanning or Hamming window $w(i)$, respectively. Once, we have obtained the STFT, the S-method (5) may be realized by the system whose architecture is presented in Fig. 2 [for a rectangular window $P(i)$ whose width is $2L_d + 1 = 9$].

IV. Architecture for Signal Dependent Realization

In the previous section, we considered the system in which the width of window $P(i)$ is not dependent of the signal. Theoretically, this width should be such that the summation over all nonzero values of the STFT is performed in (5), for each signal component. For example, if the STFT of each signal component, for a given instant n, is M samples wide, then the window $P(i)$ width should be $2L_d + 1 \geq M$. The optimal value, with respect to the calculation complexity [3], cross-terms elimination [3]–[8], and noise influence [9], is the smallest possible one producing the auto term shape the same as the WD does, i.e., $L_d = (M-1)/2$. However, if the time-frequency analysis is performed for a multicomponent signal, whose widths of the STFT's components are different and equal $M_1, M_2, \cdots, M_p$, then the signal independent window $P(i)$ should have the width $2L_d + 1 = M_{\max} = \max\{M_1, M_2, \cdots, M_p\}$. But, for the entire time-frequency plane $(n,\,k)$, except at the central points of the widest component, the window $P(i)$ will be overlong. This will have a negative influence to the time-frequency representation, with respect to the previous three essential aspects (noise, cross-terms, and calculation complexity). These are the reasons why we will introduce the system with signal dependent window $P(i)$ width, denoted by $2L_d(k) + 1$. The value of $L_d(k)$ should follow the

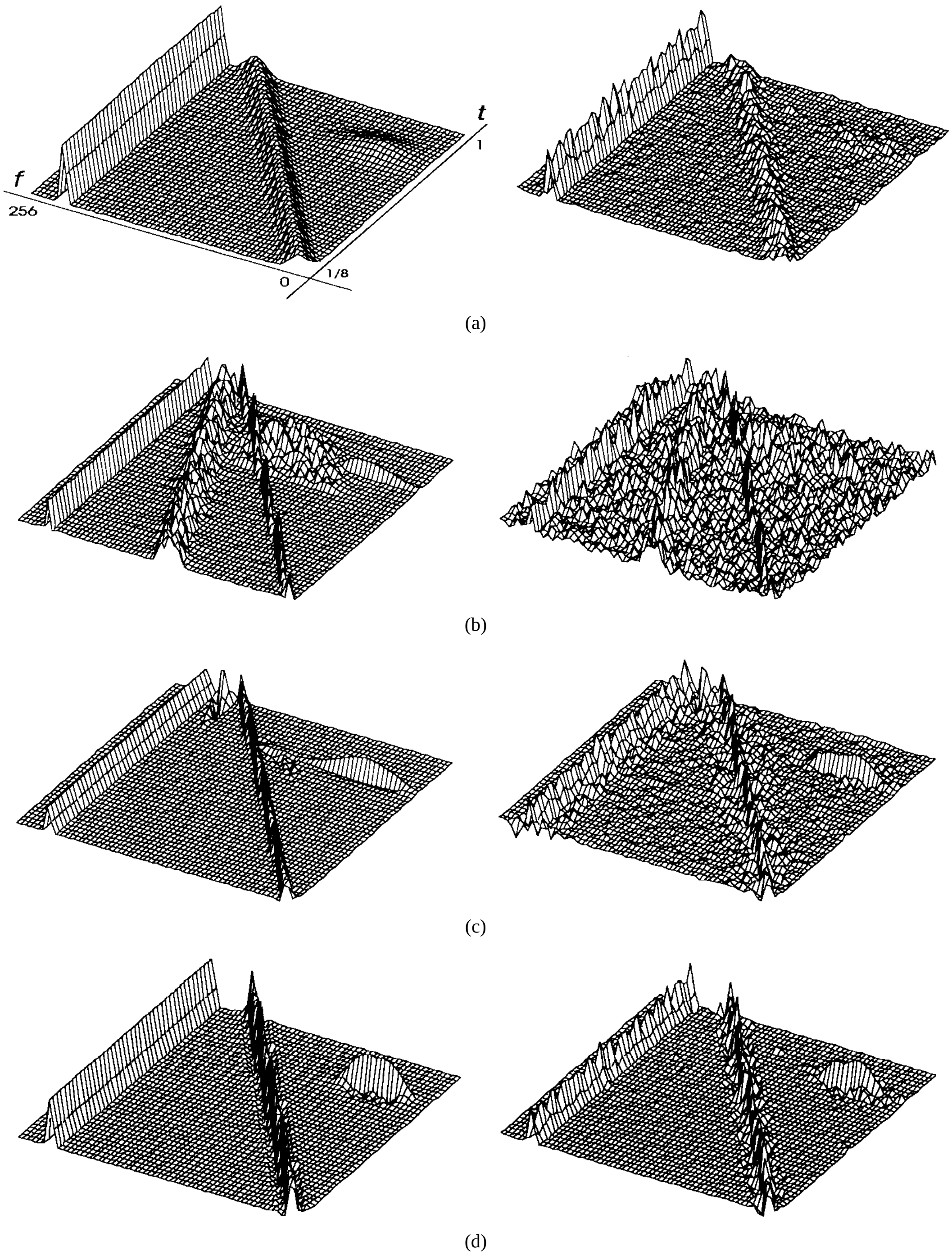

Fig. 6. Time-frequency representation of a multicomponent signal. (a) Spectrogram. (b) Wigner distribution. (c) Signal independent S-method. (d) Signal dependent S-method. Noisy signal representations are given on the right-hand side.

widths of the STFT of the signal's components. It should include the summation in (5) over the terms where $\mathrm{STFT}\,(n, k+i)$ and $\mathrm{STFT}^*\,(n, k-i)$ are different from zero. But, the variable width of $P(i)$ should exclude the summation where one or both of the previous components are equal to zero, and in addition, it should stop the summation outside a component. In this way we will not pick up the noise by summation over the points that are not needed with respect to the signal presentation quality. Also, the cross terms, between nonoverlapping components in the time-frequency plane, will be completely avoided. Obviously, the number of the numerical operations will be decreased with respect to the constant window $P(i)$ width, as well. For example, if we apply this method to a monocomponent signal with $|\mathrm{STFT}\,(n, k)| > 0$ only for $|k - k_0| \leq 2$ and a given n, then the window $P(i)$ should have the width such that $L_d(k) = 0$ for all k, except $L_d(k_0) = 2$ and $L_d(k_0 \pm 1) = 1$.

The variable window $P(i)$ width will be realized using logic circuits that will turn off all lines $i \geq i_0$ for a given k when any $\mathrm{STFT}\,(n, k+i_0)$ or $\mathrm{STFT}^*\,(n, k-i_0)$ are equal to zero or, in practical applications, less than an assumed reference value R_n (index $_n$ is to indicate that this level will, in general, be time n dependent). We defined the reference value as a fraction of the maximal value of $|\mathrm{STFT}\,(n, k)|$ for all k and a given n, i.e., $R_n = \max_k |\mathrm{STFT}\,(n, k)|/Q$, where $1 \leq Q < \infty$. Obviously, for $Q = 1$ the SPEC will be obtained, while $Q \to \infty$ will produce the WD (in the example, which follows, we used $Q = 5$). The logic circuits should provide such control signals that will allow summation in (5), up to the $L_d(k)$. Let us denote by $x_i (i = 0, 1, \cdots, N-1)$ the outputs from the comparator circuit, Fig. 3:

$$x_i = \begin{cases} 1 & \text{for } |\text{STFT}\,(n,i)| > R_n \\ 0 & \text{for } |\text{STFT}\,(n,\, i)| \le R_n \end{cases}. \tag{7}$$

Switchers' control signals (which will or will not allow the ith summation in (5), for a given k, Fig. 4) are defined by

$$c_i = \prod_{m=1}^{i} x_{k+m} x_{k-m} \qquad \text{for } i = 1, 2, \cdots, L_{d\max} \tag{8}$$

where we assumed (without loss of generality) that the maximal possible window $P(i)$ width is $2L_{d\max} + 1$. The spectrogram value will be forwarded to the output, even if $|\text{STFT}\,(n,k)| \le R_n$ for all k, so $c_0 \equiv 1$. The architecture of logic circuits that will produce the control signals, is presented in Fig. 3. Its position, within the entire system, is given in Fig. 5.

In the case of a high noise, the value $x_i = 0$ ($|\text{STFT}\,(n,i)| \le R_n$) may occur even if nonnoisy value $|\text{STFT}\,(n,\, k)|$ is greater than the reference value. In this case, we may just disregard this, being aware that the value of spectrogram is forwarded to the output anyway (as we did in the example), or to introduce a slightly more complicated logic function which will stop the summation in (5) only after two subsequent zeros of x_i and x_{i+1} are detected. In the second case, control signals c_i are defined by an expression formally the same as (8) with x_m being replaced by $\chi_m = x_m + x_{m+1}$, where "+" denotes a logical OR operation. This form may be used not only in the case of noisy signals, but also if the STFT may assume zero (or less than reference R_n) value within a single auto term.

The system presented here may be directly applied for the L-Wigner distribution [4]–[6], [15] realizations.

V. Example

Consider a multicomponent noisy signal:

$$\begin{aligned} x(t) &= f(t) + n(t) \\ &= e^{j1400t} + e^{j680(t-0.1)^2} \\ &\quad + 4e^{-[150(t-0.8)]^2} e^{j187.5t^2} + n(t). \end{aligned} \tag{9}$$

The variance of a Gaussian white noise $n(t)$ is $\sigma_n^2 = \frac{1}{2}$. The Hanning window $w(i)w(-i)$ in the WD of the width $T = 0.25$, is used. Approximately, the Hanning window $w(i)w(-i)$ is obtained with $a_0 = 0.6366$, $a_{\pm 1} = 0.2122$, $a_{\pm 2} = -0.042\,44$ in Fig. 1. Signal is sampled at T/N in the STFT and the S-method, while at $T/(2N)$ in the WD, with $N = 128$. The SPEC of signal (9) is shown in Fig. 6(a). One may observe that: 1) All distribution components are spread in the time-frequency plane, except the one representing pure sinusoid; 2) The cross terms do not exist; 3) The noise is especially exhibited just in the region where the spectrogram is different from zero. In the WD, Fig. 6(b), all components are highly concentrated, but the cross terms are present, as well as the noise over the entire time-frequency plane. The S-method, Fig. 6(c), produces the same signal representation as the WD does, but the cross terms are significantly reduced (they appear only in a tiny region when two components are very close) and the noise influence is decreased. Its significant presence may be spotted only in the region around the signal's components, defined by $2L_d + 1 = 9$. Complete elimination of the cross terms (for nonoverlapping components), as well as further reduction of the noise, is achieved by the variable (and self-adaptive) window $P(i)$ width in the S-method, Fig. 6(d) Reference value $R_n = \max_k |\text{STFT}\ (n,\, k)|/5$ and $L_{d\max} = 4$ are used. An application of the proposed method to the time-frequency analysis of real seismic signals is presented in [18] and [19].

VI. Conclusion

The systems for signal independent and signal dependent time-frequency analysis are presented. These systems produce better time-frequency signal representation than the spectrogram and the Wigner distribution, regarding to the most essential aspects, such as noise influence, cross-terms, over sampling and calculation complexity.

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