filter and -51 dB for a complex lowpass filter. This is an additional gain of 3 and 6 dB for the two cases, respectively.

To further illustrate the ability of the proposed algorithm to decompose a wavelet, we have applied the algorithm to a wavelet generated by the Daubechies $h2$ wavelet (its four filter coefficients are 0.341 506 350 946 11, 0.591 506 350 946 11, 0.158 493 649 053 89, and $-0.091\,506\,350\,946\,11$) [9]. The lengths of both the bandpass and lowpass filters were set to 4. The initial lowpass filter setting was $[1, 2, 1, 0]^T/4$. The convergence curve is shown in Fig. 1(b). The error is as high as 0 dB at the beginning. It decreases as adaptation proceeds and reaches a minimum value of -182 dB in about 150 iterations. The nonzero final error value is due to finite numerical accuracy in the simulation. The convergence in this case is exponential. The convergence characteristics of the mean-square coefficient errors of the two filters are very close to the curve shown in Fig. 1(b). It is clear that the proposed algorithm is effective and converges to the desired optimum solution.

IV. Conclusions

We have proposed an iterative method to decompose a mother wavelet into a bandpass and lowpass filter pair according to the two-scale relationship. The method computes the LS solutions of the two filters in alternate iterations until convergence. The convergence and validity of the method were confirmed by simulations.

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A Note on "An Overview of Aliasing Errors in Discrete-Time Formulations of Time-Frequency Representations"

LJubiša Stanković and Igor Djurović

***Abstract*—Various realizations asserting to produce the alias-free discrete-time Wigner distribution have been reviewed and analyzed by Costa and Boudreaux-Bartels. Here, a simple alias-free form of the pseudo-Wigner distribution is proposed. It is obtained from the fact that the sampled signal can formally be considered as a multicomponent signal.**

***Index Terms*—Fourier transforms, signal sampling, time–frequency analysis, Wigner distribution.**

I. Introduction

In the traditional ways of Wigner distribution (WD) computing, the signal has to be oversampled by a factor of two in order to avoid aliasing [2]. Significant efforts have been made in the investigation of possibilities for nonaliased discrete-time WD calculation of a signal sampled according to the Nyquist rate [3]–[9]. Several authors have tried to formulate nonaliased reduced interference distributions [10]–[13]. An excellent overview of these efforts and results has been done in [1] and [14]. Common to all of them, including the one presented here, is that the signal samples taken with the Nyquist rate completely determine bandlimited signal values at all other points. Thus, oversampling can be avoided by incorporating an implicit or explicit interpolation in the mathematical procedure of the Wigner distribution calculation.

In this short note, we will show that it is possible to calculate the nonaliased pseudo WD of a signal sampled according to the Nyquist rate by using the short-time Fourier transform (STFT) as a basic step. This procedure can be, in a straightforward manner, extended to the nonaliased realizations of the higher order time–frequency distributions [15]–[19].

II. Definitions and Theory

A definition of the STFT is

$$\mathrm{STFT}_a(t, \omega_a) = \int_{-\infty}^{\infty} f(t+\tau) w^*(\tau) e^{-j\omega_a \tau}\, d\tau \tag{1}$$

where index a denotes quantities in analog domain. Consider signal $f(t+\tau)$ sampled at $\tau = nT$, with $T \leq \pi/\omega_m$, where ω_m is the maximal frequency in the STFT: $\mathrm{STFT}(t, \omega_a) = 0$ for $|\omega_a| \geq \omega_m$ for any t.

The STFT of a discrete-time signal, which is denoted by $\mathrm{STFT}(t, \omega)$, is periodic in frequency $\omega = \omega_a T$ with period 2π

$$\mathrm{STFT}(t, \omega) = \sum_{k=-\infty}^{\infty} \mathrm{STFT}_a(t, \omega_a T + 2k\pi). \tag{2}$$

Thus, the STFT of a discrete-time signal could be formally considered to be a multicomponent signal with an infinite number of components

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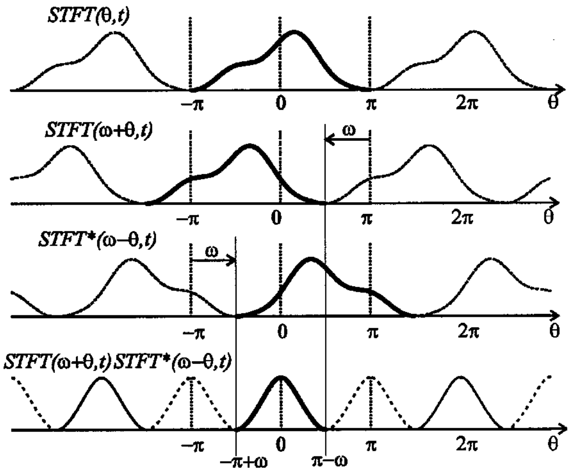

Fig. 1. Illustration of the alias-free pseudo-Wigner distribution calculation.

shifted in frequency for $2k\pi$, where k is an integer. With the sampling theorem condition $T \le \pi/\omega_m$ being satisfied, i.e., the STFT being alias-free, we have

$$\mathrm{STFT}_a(t,\,\omega_a) = \begin{cases} \mathrm{STFT}(t,\,\omega/T), & \text{for } |\omega| \le \pi \\ 0, & \text{otherwise.} \end{cases}$$

The WD, in its pseudo form, is defined by

$$\mathrm{WD}_a(t,\,\omega_a) = \int_{-\infty}^{\infty} f\left(t+\frac{\tau}{2}\right) f^*\left(t-\frac{\tau}{2}\right) w^*\left(\frac{\tau}{2}\right) w\left(-\frac{\tau}{2}\right) e^{-j\omega_a\tau}\,d\tau. \quad (3)$$

The relationship between (1) and (3) has been derived in [15] as

$$\mathrm{WD}_a(t,\,\omega_a) = \frac{1}{\pi}\int_{-\infty}^{\infty} \mathrm{STFT}_a(t,\,\omega_a+\theta_a)\mathrm{STFT}_a^*(t,\,\omega_a-\theta_a)\,d\theta_a. \quad (4)$$

The discrete domain form of (4) is

$$\mathrm{WD}(t,\,\omega) = \frac{1}{\pi T}\int_{-\pi}^{\pi} \mathrm{STFT}(t,\,\omega+\theta)\mathrm{STFT}^*(t,\,\omega-\theta)\,d\theta. \quad (5)$$

The alias-free version of the pseudo-WD (5), which is denoted with index f, can be calculated by using only the basic period $-\pi \le \omega < \pi$ of $\mathrm{STFT}(t,\,\omega)$

$$\mathrm{WD}_f(t,\,\omega) = \frac{1}{\pi T}\int_{-\pi}^{\pi} P(\omega,\,\theta)\mathrm{STFT}(t,\,\omega+\theta)\mathrm{STFT}^*(t,\,\omega-\theta)\,d\theta \quad (6)$$

where

$$P(\omega,\,\theta) = \begin{cases} 1, & \text{for } 0 \le |\theta| < \pi - |\omega| \\ 0, & \text{otherwise.} \end{cases}$$

The integration in (6) for a given ω is performed until either $-\pi \le \omega+\theta < \pi$ or $-\pi \le \omega-\theta < \pi$ is violated, i.e., for $-\pi+|\omega| \le \theta < \pi-|\omega|$; see Fig. 1. In this way, the integration over the basic period of $\mathrm{STFT}(t,\,\omega)$ [i.e., over $\mathrm{STFT}_a(t,\,\omega_a)$] is completely performed, whereas the integration over other periods $\mathrm{STFT}(t,\,\omega+2\pi)$ and $\mathrm{STFT}(t,\,\omega-2\pi)$, which were introduced by the signal discretization in time (causing aliasing in the discrete-time pseudo WD), is completely avoided; again, see Fig. 1. Distribution (6) has a form of the method from [15]–[19].

For the real signals, cross-terms between positive and negative frequencies can completely be avoided, whereas the integration over positive frequencies is performed without an analytic signal calculation by

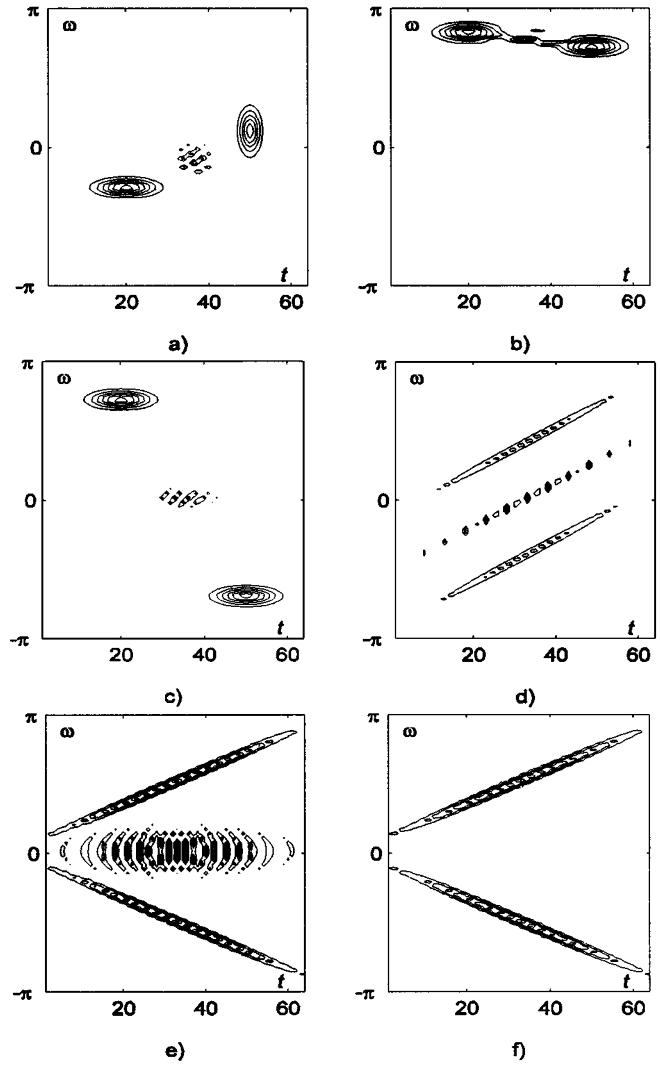

Fig. 2. (a)–(e) Alias-free pseudo-Wigner distribution for signals from [1]. (f) S-method (alias and cross-terms free pseudo Wigner distribution) of signal from Fig. 2(e).

using the integration interval in (6) defined by $0 \le \omega+\theta < \pi$ and $0 \le \omega-\theta < \pi$.

Numerical realization of the alias-free discrete-time/frequency pseudo WD is simple, according to

$$\mathrm{WD}_f(n,\,k) = \frac{1}{\pi T}\left[|\mathrm{STFT}(n,\,k)|^2 + 2\sum_{i=1}^{L_P(k)} \cdot\mathrm{Re}[\mathrm{STFT}(n,\,k+i)\mathrm{STFT}^*(n,\,k-i)]\right] \quad (7)$$

where the summation for each point $(n,\,k)$ defined by $L_P(k)$ lasts until any of the conditions ($k+i \le N$ or $k-i \ge 1$) is violated. The zero frequency component is at $k = N/2$.

Example: For all signals proposed in [1], we get a correct form of the alias-free pseudo WD; see Fig. 2.

Note that $L_P(k) = 0$ in (7) produces the spectrogram, whereas the other terms $\mathrm{Re}[\mathrm{STFT}(n,\,k+i)\mathrm{STFT}^*(n,\,k-i)]$ for $1 \le i \le L_P(k)$ improve its concentration toward the pseudo WD. Taking only few of these correction terms, we can achieve the auto-terms approximately the same as in the WD while avoiding cross-terms (the S-method [15]). In Fig. 2(f), the value of $L_P(k)$ is limited to a maximal number of six terms in (7), i.e., if $L_P(k) \geq 6$, then $L_P(k) = 6$. The choice of $L_P(k)$ in the S-method is discussed in detail in [16] and [17]. The same procedure may be used for the alias (and cross-terms)-free realization of the higher order time–frequency representations: Polynomial Wigner–Ville distributions [18], [20] and L-Wigner distributions [19].

III. Conclusion

A simple alias-free discrete-time pseudo-Wigner distribution realization is presented. The procedure may be easily extended to the alias-free realizations of some higher order time–frequency representations.

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