Frequency Based Window Width Optimization for S-transform

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Abstract— **The S-transform combines properties of the short-time Fourier (STFT) and wavelet transforms. It preserves the phase information of a signal as in the STFT, while providing the variable resolution as in the wavelet transform. However, the S-transform suffers from poor energy concentration for some classes of signals. A modification to the existing S-transform is proposed in this paper to enhance the energy concentration in the time-frequency domain. An improvement is achieved by introducing an additional parameter which can be used to optimize the window width. The optimization is performed over frequency and the proposed modification keeps the frequency marginal of the S-transform. The proposed scheme is tested on a set of synthetic signals. The results show that the proposed algorithm produces enhanced energy concentration in comparison to the standard S-transform. Also, the results show that for various signal types the proposed algorithm achieves higher signal concentration in comparison to other standard time-frequency transforms, such as, STFT and pseudo Wigner-Ville distribution. Furthermore, it is concluded by numerical study that the proposed algorithm provides more accurate estimation of the instantaneous frequency than the standard S-transform.**

I. Introduction

The time-frequency analysis of a signal depicts variation of the signal's spectrum with time. In an ideal case, the time-frequency representation provides only information about the frequency occurring at a given instant without cross-information about adjacent instants [1] [2]. Therefore, the main objective of time-frequency distribution functions is to be as close as possible to the ideal time-frequency representations, that is, to obtain time-varying spectral density function with high resolution, and to overcome any interference if exist [3]. Hence, the energy concentration in the time-frequency domain is one of its very important and intensively studied aspects in time-frequency analysis [4].

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In recent years, many different algorithms have been proposed to obtain the time-frequency representations of a signal. The recently proposed S-transform is a combination of short-time Fourier (STFT) and wavelet transforms, since it employs a variable window length and the Fourier kernel [5]. The advantage of S-transform is that it preserves the phase information of signal, and also provides a variable resolution similar to the wavelet tilings. In addition, the S-transform is a linear transform that can be used as both an analysis and a synthesis tool, which is not the case with some of the bilinear transforms such as Wigner-Ville distribution. However, the S-transform suffers from poor energy concentration in some situations. Especially poor energy concentration is achieved at the higher frequencies (the terms low and high frequencies are loosely used, and they describe whether the particular frequency is close to the highest analyzed frequency or it is much lower). For example, let's assume that a linearly increasing FM signal (i.e. a chirp) is being analyzed. At the high frequency, the energy concentration around the instantaneous frequency of the chirp is very poor, since the S-transform provides excellent time localization at higher frequencies, but poor frequency localization.

The major contribution of this paper is an optimization of the window width used in the S-transform through an additional parameter in the transform, which will enhance the energy concentration around the component's instantaneous frequency. The optimization is performed for each frequency resulting in the modified S-transform with preserved frequency marginal property.

The proposed scheme has been tested on a set of synthetic signals. The time-frequency representations obtained by the proposed algorithm are compared with those obtained from the standard S-transform. The results have shown that the proposed algorithm enhances the energy localization of the signals. Also, the results show that for some signal classes, the proposed algorithm is capable of achieving higher concentration than other standard techniques, such as, the short-time Fourier transform and pseudo Wigner-Ville distribution. Furthermore, the results show that the proposed algorithm is more accurate than the standard S-transform for the instantaneous frequency estimation. This can be of significant importance in many applications where precise energy concentration is desirable, such as communications and time series analysis.

This paper is organized as follows. In Section II, a review of the standard S-transform is given along with the modified form of the S-transform. Details of the optimization process for the proposed algorithm are covered in Section III. Section IV evaluates the performance of the proposed scheme using synthetic signals. Numerical study of the proposed transform as the instantaneous frequency estimator is also given in this section. Conclusions are drawn in Section V.

II. Background Theory

The standard S-transform of a signal $x(t)$ is given by a convolution integral as [5]:

$$S_x(t,f) = = \int_{-\infty}^{+\infty} x(\tau) w(t-\tau, f) e^{-j2\pi f\tau} d\tau = \int_{-\infty}^{+\infty} x(\tau) \frac{1}{\sigma(f)\sqrt{2\pi}} e^{-\frac{(t-\tau)^2}{2\sigma(f)^2}} e^{-j2\pi f\tau} d\tau \quad (1)$$

The advantage of the S-transform over the STFT is that the standard deviation σ is a function of frequency, f, defined as

$$\sigma(f) = \frac{1}{|f|}. \quad (2)$$

In other words, the window function is a function of time and frequency. Width of the window is determined by frequency. It can be seen that the window is wider in the time-domain for lower, and narrower for higher frequencies. In other words, the window provides good localization in the frequency-domain for low frequencies, while it provides good localization in the time-domain for higher frequencies. It is clear that the time-frequency atoms for the S-transform are arranged in the same way as for the wavelet transform.

An important property of the S-transform is that its integral over time is equal to the Fourier transform:

$$X(f) = \int_{-\infty}^{+\infty} S_x(t,f) dt \quad (3)$$

where $X(f)$ is a Fourier transform of the signal, $x(t)$. This is an important property, since it distinguishes the S-transform from the wavelet transform [5], and also it represents a form of the frequency marginal property [6].

The disadvantage of the S-transform is the same assignment of the standard deviation for all signal components at all frequencies. That is, σ, is always defined as a reciprocal of the frequency. Some signals would benefit from different value of the standard deviation for the window function. A simple, but powerful improvement to the S-transform can be made by defining the standard deviation of the window as:

$$\sigma(f) = \frac{1}{|f|^p} \quad (4)$$

resulting in the modified S-transform given as:

$$S_x^p(t,f) = \frac{|f|^p}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} x(\tau) e^{-\frac{(t-\tau)^2 f^{2p}}{2}} e^{-j2\pi f\tau} d\tau. \quad (5)$$

where the new parameter p controls the width of the window. If an appropriate value of p can be determined for a given signal, an improved time-frequency localization becomes possible. It can be shown that the proposed modification of the S-transform keeps the frequency marginal property:

$$X(f) = \frac{1}{W(0,f)} \int_{-\infty}^{+\infty} S_x^p(t,f) dt \quad (6)$$

where $W(\alpha, f)$ is a Fourier transform (from t to α) of the window function, $w(t,f)$.

III. Proposed Optimization Algorithm

The proposed implementation of the S-transform provides us with the additional parameter, p, which can be adjusted to achieve the time-frequency representation of the signal with improved energy concentration. It is desired to develop a scheme that can be used to obtain the optimal value of the parameter p for a given signal. In the time-frequency analysis, the aim of the optimization is to obtain a time-frequency representation as close as possible to the ideal one, i.e., signal components most closely resemble the instantaneous frequency.

Therefore, the optimal value of p will be found based on the variation of the concentration measure proposed in [7]. The measure is designed to minimize the energy concentration for any time-frequency representation based on the automatic determination of some time-frequency distribution parameter. In this case, the objective is to minimize the energy concentration through the parameter p which controls the window width.

The optimization can be performed either with respect to time or to frequency. In order to understand the outcome of either approach, it is critical to understand the marginals of the S-transform. The time and frequency marginals are important quantities in time-frequency analysis since they ensure that the spectral, temporal, and total signal energies are accurately reflected in the time-frequency domain. They also assure that global quantities such as mean time, mean frequency, duration, and bandwidth are correctly given [8]. As shown by (3), the S-transform has well-defined frequency marginal. However, the S-transform does not satisfy the time-marginal property. This implies that in a case of a signal with a time-varying amplitude, integrating over the frequency in the time-frequency domain would destroy the information about the signal's amplitude, while integration over the time preserves information on the spectral content[1]. Therefore, it is a natural choice to perform the optimization in the frequency-domain. In this way, the frequency marginal of the S-transform can be preserved, and information about the amplitude of spectral components is not destroyed.

For the above reasons, a scheme for finding the optimal variation of the parameter based on frequency is given below. The scheme provides us with the frequency-varying parameter, $p(f)$, which is used in the proposed S-transform modification. Therefore, the algorithm for $p(f)$ can be summarized through the following steps.

1. For p selected from the set $p \in (0, 1]$, compute S-transform of the signal, $S_x^p(t, f)$ according to (5).
2. For each p and a frequency f compute

$$CM(f,p) = \frac{1}{\int_{-\infty}^{+\infty} |S_x^p(t,f)|^q \, dt} \tag{7}$$

where $q \in (0, 0.25]$.
3. Optimal value of p for the considered frequency f maximizes concentration measure $CM(f,p)$

$$p_{opt}(f) = \arg\max_p [CM(f,p)] . \tag{8}$$

4. Set the modified S-transform as:

$$S_x^a(t,f) = S_x^{p_{opt}(f)}(t,f). \tag{9}$$

The algorithm specifies that the parameter p should not exceed 1. The reason is that for $p \gg 1$ the window function becomes very narrow in the time-domain, and for very high value of p it approaches the Dirac function. Very narrow window in the time-domain is suitable only for analysis of the Dirac function itself, or a sum of the Dirac functions.

All time-frequency representations obtained in step 1 satisfy the frequency marginal, and for a given frequency, the representation, which provides the best localization in the time-frequency domain, is chosen in the step 2. Parameter q in step 2 is selected to be within $q \in (0, 0.25]$. Namely, parameter $q = 0$ would produce $\int_{-\infty}^{\infty} |S_x^p(t,f)|^q \, dt$ equal to time interval in which component is detected for given frequency. This time-interval could be caused by actual component on this

[1] We developed a technique for optimization of p for each instant that requires elaborate normalization strategy. This technique will be reported elsewhere.

frequency but also by spread of components from adjacent frequency bins. Then the measure $CM(f,p)$ would be maximal if influence of adjacent frequency bins is minimized (value $\int_{-\infty}^{\infty} |S_x^p(t,f)|^q\, df$ minimal). However, due to the potential presence of noise, sidelobes of the signal and other effects, it is not a good idea to use $q = 0$. Instead we propose to use a q in the range $q \in (0, 0.25]$. Note that using larger values of q, namely $q \approx 1$, could not produce appropriate results since value $\int_{-\infty}^{\infty} |S_x^p(t,f)|^q\, dt$ approaches toward constant (toward marginal property) for each p. Then it would be difficult to highlight any difference among the time-frequency representations for different p.

In an actual implementation of the algorithm, it might be beneficial to use some of the available signal processing tools to reduce the effects of the discrete implementation. Such tools include zero-padding the signal or windowing the signal before proceeding to the step 2. This is especially critical for wideband signals, which might show some irregular properties unless they are properly windowed [9]. A wide selection of the windows is available in the literature and usually Gaussian or Kaiser windows are used to smooth the edges of the finite duration signals, due to the fact that they have a parameter to control the width of the window.

While discussing the implementation of the proposed scheme it is important to mention that in our simulations only one global optimum point for p or q exists and the local minimums do not exist. Therefore, a least mean square based technique could be developed in order to diminish computation complexity [7].

As a last remark, it should be mentioned that further improvements to the concentration of the S-transform could be achieved by introducing a method for the time-frequency analysis similar to an S-method or its signal adaptive variation [10], which is developed based on the relation between the Wigner-Ville distribution and a spectrogram.

IV. Simulation Results and Discussion

In this section, the performance of the proposed algorithm for the S-transform is examined using a set of synthetic test signals. The goal is to examine how the proposed algorithm performs in comparison to the standard S-transform. Furthermore, a comparison with other standard time-frequency techniques, such as the short-time Fourier transform (STFT) and the pseudo Wigner-Ville distribution (PWVD) is given. In the actual implementation, the sampling period used is $T = 1/256$ seconds, and the set of p values is selected by $p = \{0.01n : n \in \mathbb{N}$ and $1 \le n \le 100\}$. For the STFT and the PWVD, a Gaussian window is used with standard deviations equal to 0.05 and 0.1, respectively.

Let's begin the performance analysis of the algorithm with the following signal:

$$x_1(t) = \cos(68\pi t - 20\pi t^2) + \cos(2\pi \sin(5\pi t) + 120\pi t) + \cos(168\pi t + 28\pi t^2) \quad (10)$$

where $0 \le t < 1$, and the signal does not exist outside the interval. By examining the time-frequency representations of the signal, it can be noticed that the STFT is capable of concentrating the two chirps well, but the concentration of the sinusoidally modulated component is poor.

The representation obtained by the PWVD contains significant cross-terms. The standard S-transform suffers from the same problem as the wavelet transform, that is, at the low frequencies it has good frequency resolution, but as the frequency increases, the frequency resolution deteriorates. This is evident from Fig. 1(d). With the proposed algorithm more uniform resolution is obtained at both low and high frequencies as shown in Fig. 1(e). It is worthwhile to consider the behavior of $p(f)$, which is shown in Fig. 1(f). It is apparent that as the frequency increases, the value of p decreases, which means that the window is becoming wider in the time-domain. A wider window in the time-domain provides better localization in the frequency-domain. Therefore, this indeed enhances the time-frequency representation of the signal in comparison to the standard S-transform, and resulting in better concentrated time-frequency representation.

In the first example, the signal with linearly increasing/decreasing frequencies along with a sinusoidally modulated frequency component

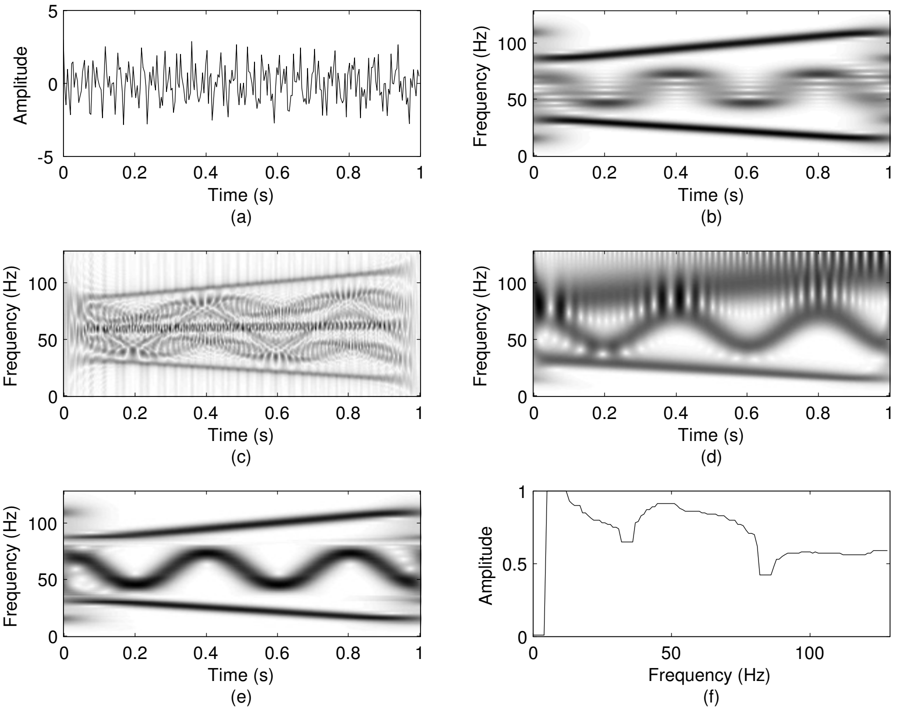

Fig. 1. Test signal $x_1(t)$: (a) Time-domain representation; (b) STFT; (c) PWVD; (d) standard S-transform; (e) $S_x^{p(f)}(t, f)$; (f) optimal variation of parameter p across the frequencies, $p(f)$.

is examined. It is worthwhile to examine how the algorithm behaves for a multicomponent signal with the crossing components and parabolic instantaneous frequency:

$$x_2(t) = \cos(144\pi(t-0.3)^3 + 30\pi t) + \cos(128\pi t - 50\pi t^2) \quad (11)$$

where $0 \leq t < 1$, and the signal does not exist outside the given interval. The signal is examined again by the STFT, the PWVD, the standard S-transform and the proposed scheme.

Even though, the representation obtained by the PWVD, shows that the components are well concentrated in the time-frequency domain, the presence of the cross-terms diminishes readability of the representation. The STFT is capable of concentrating the chirp, however, the concentration of the parabolic component is very poor, especially, at the higher frequencies. The standard S-transform provides higher concentration of the parabolic component than the STFT, but the concentration of the linear chirp is poor at the higher frequencies. From Fig. 2(e), it is obvious that the proposed algorithm for the S-transform is able to diminish the spectral leakage which is occurring with the standard S-transform. It also provides good concentration of the parabolic component. By examining the behavior of $p(f)$ shown in Fig. 2(f), it can be seen that the window for the proposed algorithm becomes wider in comparison to the window in standard S-transform, which is significant to form more concentrated time-frequency representation of the signal.

So far it has been shown that the proposed algorithm for the S-transform is capable of enhancing the energy concentration. Now, it is interesting to examine the behavior of the proposed algorithm in the presence of noise, and how noise affects the accuracy of the instantaneous frequency estimation based on the peak values [11] of $S_x^p(t, f)$ for various values of p. In

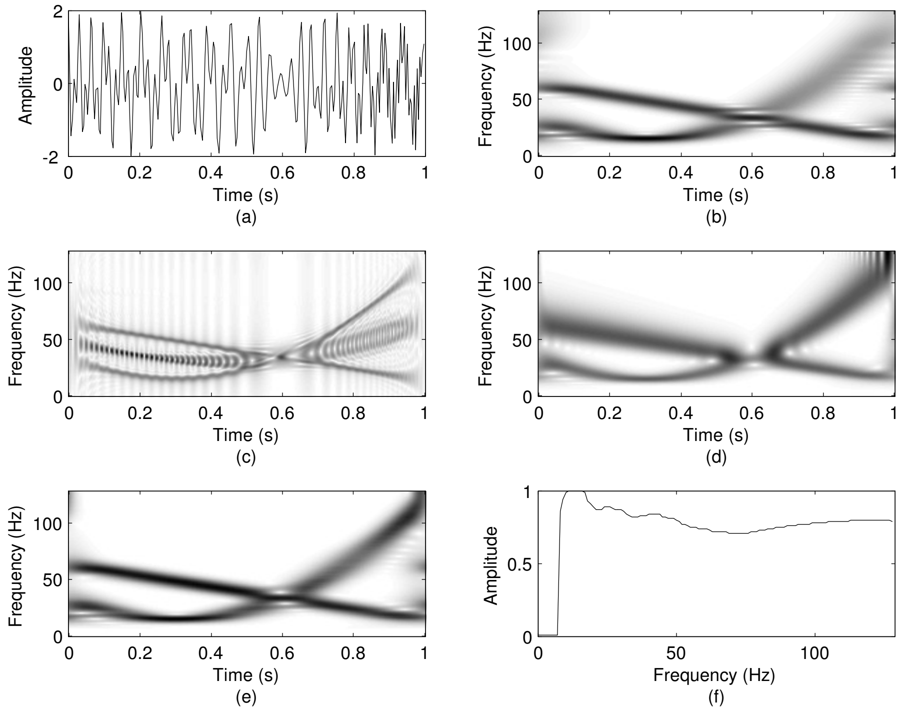

Fig. 2. Test signal $x_2(t)$: (a) Time-domain representation; (b) STFT; (c) PWVD; (d) standard S-transform; (e) $S_x^{p(f)}(t,f)$; (f) optimal variation of parameter p across the frequencies, $p(f)$.

order to examine such behavior the following signal is used:

$$x_3(t) = A\sin(100\pi t + 4\pi\cos(4\pi t)) \qquad (12)$$

where $0 \leq t < 1$ and $A = 1$. The signal is contaminated with an additive white Gaussian noise, and the signal to noise ratio (SNR) is given by $SNR = 10\log_{10}(A^2/\sigma_n^2)$ with A being a signal's amplitude and σ_n^2 being the variance of the noise. In order to investigate the behaviour of the proposed algorithm in the presence of noise, a range of SNR values are used, and the mean square error (MSE) is evaluated for the instantaneous frequency estimation of the given sinusoidally modulated signal. The estimation is performed based on the peaks of the $S_x^p(t,f)$ for various values of p, namely, $p = 0.1$, $p = 0.3$, $p = 0.5$, $p = 0.7$, $p = 1$ and adaptive value, $p(f)$. It is noted that $S_x^p(t,f)$ for $p = 1$ corresponds to the standard S-transform. The SNR varies from 0 dB to 15 dB in a 1-dB step. For each SNR value, 100 realizations are used.

Fig. 3 represents the results of such analysis. The horizontal axis represents the SNR (in decibels), and the vertical axis represents the MSE for the instantaneous frequency estimation. The simulation results show that the proposed algorithm yields a smaller MSE than the standard S-transform and, therefore, is more accurate in estimating the instantaneous frequency of the noisy signal. Similar situation has been observed for other test signals as well.

V. Conclusions

In this paper, a window width optimized algorithm for the S-transform is developed, which provides enhanced energy concentration of the signal in the time-frequency domain. The improvement is performed through an additional parameter p, which controls the width of window used in the S-transform. The parameter is determined through the optimization procedure that maximizes the energy concen-

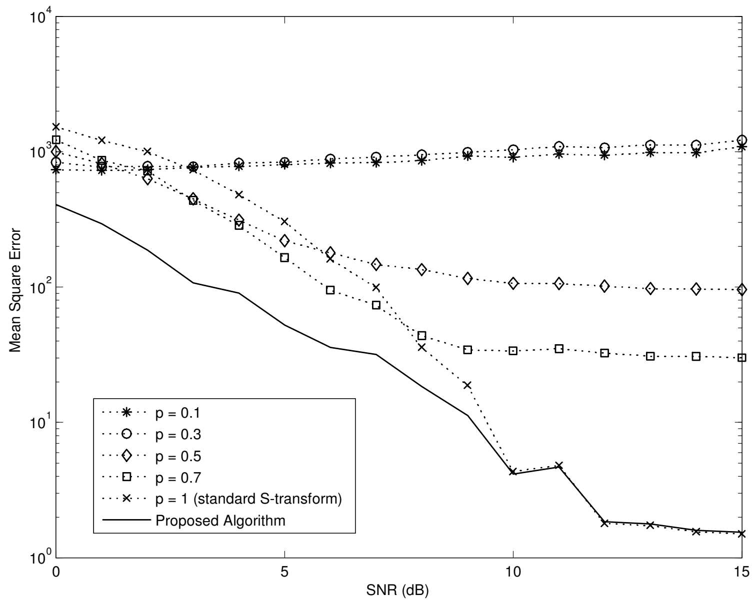

Fig. 3. MSE for the instantaneous frequency estimation for $S_x^p(t, f)$ based on various values of p.

tration of the time-frequency representation for each frequency bin. The proposed scheme is evaluated numerically and compared with the standard S-transform by using a set of synthetic test signals. The results have shown that the proposed algorithm can significantly enhance the energy localization of the signals in comparison to the standard S-transform. Also, it is showed that the proposed algorithm is capable of achieving higher energy concentration than other standard algorithms, such as, the STFT and the PWVD. Furthermore, it has been numerically shown that the proposed algorithm provides more accurate estimation of the instantaneous frequency in comparison to the standard S-transform.

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