Viterbi Algorithm for Chirp-Rate and Instantaneous Frequency Estimation

Igor Djurović

Abstract— **An instance of the Viterbi algorithm has been applied to the cubic phase function and chirp-rate estimation. The proposed algorithm has shown excellent performance for high noise environment. The obtained chirp-rate estimate is used in the instantaneous frequency estimation. The proposed instantaneous frequency estimator gives superior performance with respect to the state-of-the-art techniques for signals with non-linear instantaneous frequency.**

I. Introduction

The Viterbi algorithm (VA) for hidden state estimation has found diverse applications in various scientific fields [1], [2]. It seems that it is best known in communication systems where it is a common tool for decoding convolution codes. Recently, it has been applied to the Wigner distribution (WD) based instantaneous frequency (IF) estimation [3]. The IF is the most important parameter of non-stationary signals. If it is available or accurately estimated, the estimation of other signal parameters can be performed on dechirped signal in the same fashion like common processing techniques of stationary signals. The time-frequency (TF) representations, of which the WD is a prominent member, are used for development of various non-parametric IF estimation tools [4], [5]. Detailed analysis of the TF-based IF estimators is given in [5]. The following sources of errors are identified in the case of the TF-based IF estimators: (a) bias introduced by non-linearity in the IF function; (b) small noise influence that can move peak of the TF representation within the signal auto-term; (c) high noise influence that can move peak of the TF representations outside of the signal auto-term; (d) errors caused by discretization of the TF grid; (e) influence of other signal components (cross-terms) etc. The mentioned instance of the VA has been primarily developed for handling the high noise influence [3]. Originally it has been applied to the WD (referred here as the VA-WD) but it has also been used for other TF representations and for some practical applications [6]. Furthermore, it has been applied on connecting components in the TF plane like in [7]. The proposed algorithm is quite robust to noise influence and the experiments have shown that it is the most robust existing technique for extremely high noise [3]. The proposed technique is able to produce accurate results for linear FM signals up to -10dB. For signals with non-linear IF the amplitude of the WD decreases, i.e., the WD spreads over TF plane. Position of the maximum is moved from the IF due to the higher order derivatives in signal phase [8] causing the estimation bias. In addition, decreasing of the WD amplitude causes reduced robustness to the noise influence. The VA-WD improves results for such signals but not in so spectacular manner as in the case of the linear FM signals.

Signal Processing, Vol. 91, No. 5, May 2011

Recently, O'Shea has proposed the cubic phase function (CPF) for the chirp-rate (CR) estimation [9]. This function is quite similar to the WD having the same order of non-linearity but it is able to estimate the CR (second derivative of the signal phase). In this paper, we have proposed the VA for the CR estimation based on the CPF (VA-CPF for short). This is straightforward extension of the algorithm from [3]. Like in [3], the proposed technique significantly improves estimation of the CR for high noise environments. Here, we have used the obtained CR estimate to perform the IF estimation in the second stage. This technique produces significantly better results for signals with non-linear IF than the original technique from [3] (it will be referred to as the VA-CPF-WD for short since the WD is used in the second step, i.e., in the IF estimation).

The letter is organized as follows. The background related to the IF and CR estimation is given in Section II. The VA-CPF-WD estimator is proposed in Section III. Numerical study is presented in Section IV with conclusions and discussions provided in Section V.

II. IF and CR estimation

Here we will consider frequency modulated (FM) signal defined as:

$$x(t) = Ae^{(j\phi(t))}. \quad (1)$$

The signal is embedded in Gaussian noise with independent real and imaginary parts $y(t) = x(t)+\nu(t)$, where $E\{\nu(t)\} = 0$ and $\text{var}\{\nu(t)\} = \sigma^2$. The IF is defined as the first derivative of the signal phase $\omega(t) = \phi'(t)$ while the second derivative of the signal phase is the CR $\Omega(t) = \phi''(t)$. The IF is extremely important since its knowledge or accurate estimation allows simple estimation of other signal parameters (phase and amplitude), accurate time-varying filtering, etc. The CR is also quite important since among other applications its knowledge can be used for focusing of SAR and ISAR images defocused by various phenomena distorting received radar signals [10], [11].

A. Wigner distribution

The TF representation are common tool for non-parametric IF estimation. The position of the TF representation maxima in the considered instant is the simplest and the most commonly used TF-based IF estimator. Here, the WD is considered

$$WD(n,k) = \sum_{k=-N/2}^{N/2-1} y(n+k)y^*(n-k)e^{(-j4\pi nk/N)} \quad (2)$$

where $y(n) = y(n\Delta t)$, Δt is the sampling rate and N is number of samples within a considered interval. The IF can be estimated as $\hat{\omega}(n) = \pi\hat{k}(n)/(\Delta t N)$ where

$$\hat{k}(n) = \arg\max_k WD(n,k). \quad (3)$$

Note that for non-noisy signal the WD is ideally concentrated on the IF for linear FM signal and it can be expected excellent performance of the WD-based IF estimator. However, for non-linear FM signal amplitude of the WD would decrease with appearance of inner interferences. These effects reduce performance of the WD as the IF estimators. Commonly reducing the bias (inner interferences) by employing the higher-order TF representations has as a drawback increased influence of noise [12].

B. Cubic phase function

The CPF is defined as:

$$C(n,\Omega) = \sum_{k=-N/2}^{N/2} y(n+k)y(n-k)e^{(-j\Omega(k\Delta t)^2)} \quad (4)$$

One may notice similarity with the WD: slightly different auto-correlation and changed complex exponential. However, the CPF is able to estimate the CR

$$\hat{\Omega}(n) = \arg\max_{\Omega} |C(n,\Omega)|. \quad (5)$$

The CPF is ideally concentrated on the CR for cubic phase signals $\phi(t) = \sum_{i=0}^{3} a_i t^i/i!$, with the IF $\omega(t) = \sum_{i=1}^{3} a_i t^{i-1}/(i-1)!$ and the CR $\Omega(t) = \sum_{i=2}^{3} a_i t^{i-2}/(i-2)! = a_2 + a_3 t$. For higher-order phase signals bias and other effect will be observed in the CR estimation [13].

III. Viterbi algorithm

A. IF estimation

An instance of the VA is developed for the IF estimation of monocomponent signals. The WD was the TF tool on which the VA has been applied. This estimator can be defined as

$$\hat{k}(n) = \arg\min_{k(n)} p(k(n); n_1, n_2) \quad (6)$$

where $p(k(n); n_1, n_2)$ is the sum of path penalty functions in the TF plane from the instant n_1 to the instant n_2 along a line $k(n)$. Commonly used path penalty functions in the VA framework are designed as logarithms of the corresponding conditional probabilities. However, this model is rather hard to be applied in the TF analysis since the conditional probabilities of TF representation states cannot be determined for general non-parametric model of FM signals and noise in an appropriate or closed form. Then two path penalty functions are used. The first one (denoted with $f()$) assumes that the IF in the considered instant is on one of the largest values of the TF representation (now it is allowed that the IF is not strictly on maximum of the TF representation like in (3)). The second one (denoted as $g()$) assumes that the changes of the IF between consecutive instants are not too large. This can be written as:

$$\hat{k}(n) = \arg\min_{k(n)} \left[\sum_{n=n_1}^{n_2-1} g(k(n), k(n+1)) + \sum_{n=n_1}^{n_2} f(WD(n, k(n))) \right]. \tag{7}$$

Functions $f()$ and $g()$ are selected in a semi-intuitive manner. Function $f()$ is formed by sorting the WD values for the considered instant and the maximal value is penalized with 0, the second largest is penalized with value 1, the third one with value 2, ... This clearly reflects the idea that even in the high noise environment the IF is on one of the highest values of the WD. The second function is set as

$$g(x,y) = \begin{cases} 0 & |x-y| \le \Delta \\ c(|x-y| - \Delta) & |x-y| > \Delta, \end{cases} \tag{8}$$

where c is weight of the penalization function while Δ is threshold above which IF variations between consecutive instants are penalized. Details on the VA-WD algorithm can be found in [3]. The realization of the algorithm is recursive using partial best paths from the previous instant for calculation of the IF estimate in the current instant.

B. Chirp rate estimation

The same algorithm can be used for the CR estimation. The CR is estimated by using the VA(VA-CR in short):

$$\hat{\Omega}(n) = \arg\min_{\Omega(n)} \left[\sum_{n=n_1}^{n_2-1} g(\Omega(n), \Omega(n+1)) + \sum_{n=n_1}^{n_2} f(|C(n, \Omega(n))|) \right]. \tag{9}$$

Functions $f()$ and $g()$ can be defined in the same manner as for the IF estimation.

C. Joint IF and CR estimation

For FM signal with cubic modulation the CPF is ideally concentrated on the CR. However, the WD is not ideally concentrated on the IF for this signal since it has inner interferences for non-linear FM function. In addition, it can be expected that for higher order polynomial FM functions the inner interferences will be smaller in the case of the CPF than for the WD. Then, the CR estimate obtained from the VA-CR is used in the IF estimation. In the first step of the algorithm the VA-CR is performed (9). Then the cumulative sum of the estimates:

$$\hat{\mu}(n) = \Delta t \sum_{k=n_1}^{n} \hat{\Omega}(k) \tag{10}$$

can be used to estimate the IF with accuracy up to the additive constant. For determination of this constant the following function is created

$$J(\mu) = \sum_{n=n_1}^{n_2} WD(n, \hat{\mu}(n) + \mu). \tag{11}$$

Function (11) is greater for lines $\hat{\mu}(n) + \mu$ that are closer to the true IF. Then the IF estimate can be calculated as:

$$\hat{k}(n) = \hat{\mu}(n) + \hat{\mu} \tag{12}$$

where

$$\hat{\mu} = \arg\max_{\mu} J(\mu). \tag{13}$$

In this way the accuracy of the IF estimation is improved since the influence of the higher order terms is reduced with respect to the VA-WD. This accuracy improvement is clearly demonstrated within the next section with three numerical examples.

IV. Numerical examples

In all numerical examples the following setup is considered: signal amplitude was $A = 1$, sampling rate was $\Delta t = 1/128$, signal was considered within the interval $t \in [-1, 1]$ with 256 samples within. The WD is evaluated with 256 samples on the frequency grid. The CR function was considered within the range of $\Omega \in [-96\pi, 96\pi]$ with 200 equidistantly spaced elements in the domain. The following IF estimators are considered: $\mathrm{WD}_{\mathrm{max}}$ - IF estimation based on the position of the WD maxima with rectangular window of 256 samples width; $\mathrm{WD}_{\mathrm{max\text{-}h}}$ - IF estimation based on the position of the WD maxima with the Hanning window of 256 samples width; $\mathrm{WD}_{\mathrm{max\text{-}med}}$ - IF estimation $\mathrm{WD}_{\mathrm{max}}$ filtered with the median filter of the width 5; $\mathrm{WD}_{\mathrm{max\text{-}h\text{-}med}}$ - IF estimation $\mathrm{WD}_{\mathrm{max\text{-}h}}$ filtered with the median filter of the width 5; $\mathrm{WD}_{\mathrm{Viterb}}$ - IF estimation with the VA-WD (WD with rectangular window) [3]; $\mathrm{WD}_{\mathrm{h\text{-}Viterb}}$ IF estimation with the VA-WD (WD with Hanning window); HO-WD - IF estimation with the form of the higher order polynomial WD (HO-WD) from [15] and Proposed - IF estimation with the VA-CPF-WD. The following CR estimators are considered: $\mathrm{CPF}_{\mathrm{max}}$ - position of the CPF maximum; $\mathrm{CPF}_{\mathrm{max\text{-}med}}$ - CPF estimation based on the position of the CPF maximum filtered with median filtering; PHAF - chirp rate estimator based on the product high order ambiguity function (PHAF) from [16], [17]; Viterbi - CR estimation with the VA-CPF.

Parameters of the both VAs (7) and (9) are the same, $c = 5$ and $\Delta = 1$.

In the first example with the cubic phase signal, the IF estimators are compared with the HO-WD from [15] and the CR estimators are compared with the PHAF [16]. Here, these techniques will be briefly reviewed.

HO-WD as an IF estimator: The HO-WD is defined as:

$$HOWD(n,k,\alpha) = \sum_{k=-N/2}^{N/2-1} y(n+k)y^*(n-k) \times e^{(-j4\pi nk/N - j\alpha k^3(\Delta t)^3)}. \quad (14)$$

Parameter α is introduced for reducing the bias and influence of the higher order terms. In our setup a set $\alpha \in [-300, 300]$ with 201 elements is used. The IF is estimated for the "best" adjusted α producing the highest value of the HO-WD (the best concentrated transform with eliminated bias)

$$\hat{k}(n) = \arg\max_k \left[\max_\alpha HOWD(n,k,\alpha)\right]. \quad (15)$$

PHAF as a CR estimator: The HAF is parametric tool for estimating parameters of polynomial phase signals [18]. For cubic phase signal with the phase $\phi(t) = \sum_{i=0}^{3} a_i t^i/i!$ the highest order coefficient $\hat{a}_3$ can be estimated as:

$$\hat{a}_3 = \frac{\arg\max_\omega |HAF(\omega;k_1,k_2)|}{24k_1k_2(\Delta t)^2}$$
$$= \frac{1}{24k_1k_2(\Delta t)^2}\arg\max_\omega |\sum_n y(n+k_1+k_2) \times y^*(n-k_1+k_2)y^*(n+k_1-k_2) \times y(n-k_1-k_2)e^{(-j\omega n)}| \quad (16)$$

where (k_1, k_2) are the lag coefficients. For reducing effects of interferences for multicomponent signals commonly the PHAF is used. Here the two HAFs with lag coefficients $(k_1^{(1)}, k_2^{(1)}) = (24, 24)$ and $(k_1^{(2)}, k_2^{(2)}) = (16, 36)$ samples are evaluated (here $k_1^{(1)}k_2^{(1)} = k_1^{(2)}k_2^{(2)}$ and frequency scaling is avoided [16]). Then the PHAF-based estimator can be written as:

$$\hat{a}_3 = \frac{1}{24k_1^{(1)}k_2^{(1)}(\Delta t)^2}\arg\max_\omega | HAF(\omega;k_1^{(1)},k_2^{(1)})HAF(\omega;k_1^{(2)},k_2^{(2)})|. \quad (17)$$

Since the chirp rate for this signal is $\Omega(t) = a_3t + a_2$ it is required also to estimate the second order coefficient a_2 that is possible after dechirping the signal of interest and performing similar PHAF-based procedure for its estimation. In our experiments it is assumed that a_2 is somehow known in advance and set $a_2 = 0$. Note that here the PHAF procedure is performed on windowed signal in the same manner as the WD and CPF are evaluated and the HAFs are interpolated with 2048 samples using the zero-padding procedure to reduce discretization errors in parameter estimation. Details on the PHAF realization can be found in [16].

Example 1. The first example is cubic phase signal $x(t) = \exp(j12\pi t^3)$. The IF for this signal is $\omega(t) = 36\pi t^2$ while the CR is $\Omega(t) = 72\pi t$. The MSE in the IF and CR estimation are given in Fig.1. It can be seen that the proposed approach significantly outperforms all the counterparts for high noise environment. Especially it is obvious for the IF estimation where the proposed technique significantly outperforms the VA-WD. The reason is in the fact that the CPF is ideally concentrated on the CR for cubic phase signals while the WD is not concentrated ideally on the IF for such signals. Then the WD has inner interferences and smaller magnitude of the maximum causing worse results in the case of the VA-WD. This is illustrated in Fig. 2. The HO-WD significantly reduces the bias in the IF estimation due to the additional term with parameter α (see Fig. 1a). However, this technique works accurately only for $SNR \geq 0$dB while the SNR threshold for the proposed technique is about SNR =-7dB what is significant improvement. Note, also that the search for the additional parameter in the HO-WD is more demanding than the VA application on the WD and the CPF. Similar, conclusions can be drawn from comparison between the proposed technique and the PHAF-based CR estimator. It is obvious that for a high SNR the PHAF based method is better than the proposed one but the PHAF method has SNR threshold about SNR =3dB while the threshold of the proposed technique is $SNR = -4$dB, i.e., 7dB lower. Commonly it is assumed than any additional auto-correlation reduces the threshold for 3dB while the VA algorithm improves the threshold for remaining 4dB.

Note that the VA algorithm is universal tool for similar problems in non-parametric estimation and that it can be applied to both the HO-WD and the PHAF but this topic remains out of this paper scope.

A single realization of the noisy process with SNR =-7dB is considered in Fig. 2. The CR estimates are depicted in Fig. 2a. It can be seen that the proposed algorithm outperforms direct estimation using position of the maximum even after filtering of the estimate is performed using the median filter. Based on this estimate the IF estimate that has approximately parabolic shape is created. However, the exact position of the this parabola in the TF plane is not known. Then search is performed using (11)-(13) in the TF plane. It can be seen from Fig. 2c that the exact position of the IF is quite difficult to be observed due to high impact of noise. Function $J(\mu)$ is given in Fig. 2d and its maximal value determines IF estimate (depicted with solid line in Fig. 2c). Other values of $J(\mu)$ correspond to the parallel parabola in the range depicted in Fig. 2c with two dotted parabolas. Note that the bias in the WD limits accuracy of the considered technique.

Example 2. In this example it is shown that for linear FM signal the proposed VA-CPF-WD produces the same accuracy as the VA-WD in the IF estimation, while providing the CR estimation as an additional feature. A linear FM signal $x(t) = \exp(j24\pi t^2)$ is considered with the IF $\omega(t) = 48\pi t$ and the CR $\Omega(t) = 48\pi$. The MSE for the IF and the CR estimation is depicted in Fig. 3. Here, it can be noticed that the proposed algorithm gives accuracy similar to the previous algorithm form [3]. However, for a high SNR the proposed algorithm produces higher MSE. This inaccuracy is caused by both discretization of the CPF and WD. Anyway, the high SNR case is not of particular interest for the VA-type of the IF estimators since they are primarily developed for a low (or extremely low) SNR.

Example 3. In this example it is considered the third and the most difficult case for the proposed algorithm with signal with non-polynomial modulation. The considered signal is $x(t) = \exp(j72\cos(\pi t/3))$. The IF and CR for this signal are $\omega(t) = -24\pi \sin(\pi t/3)$ and $\Omega(t) = -8\pi^2 \cos(\pi t/3)$, respectively. The CPF is not ideally concentrated on the CR since this signal has no linear CR. Then, the CPF-based CR estimator produces bias that propagates to the IF estimation in the second stage. Additional bias is produced with the WD since

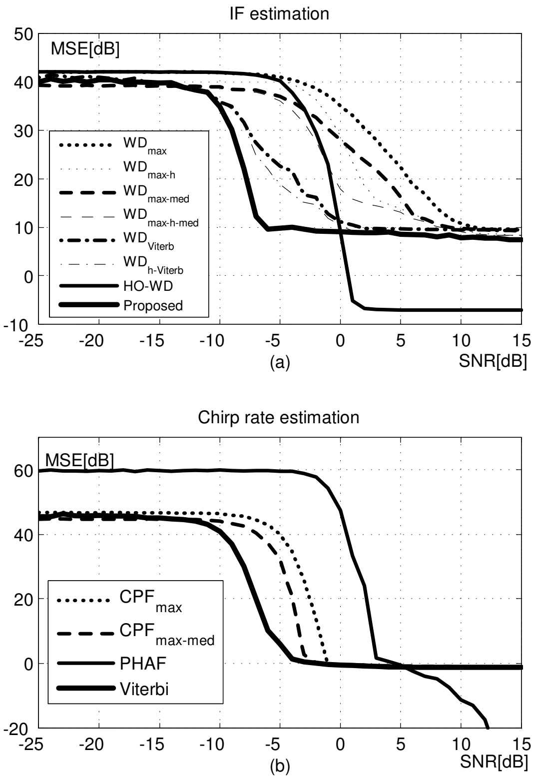

Fig. 1. Signal with cubic phase: (a) MSE for IF estimation; (b) MSE for CR estimation.

it is also biased for such signal. As it can be seen from Fig. 4, the proposed VA-CPF-WD technique gives better results than the existing approaches. Since it is examined which effect is more dangerous for accuracy of the proposed scheme: the bias introduced by higher order phase derivatives or noise, a refined set of experiments is performed. Namely, results are given for both rectangular and smoothed window (Hanning) in both WD and CPF. The aim of the smooth window is to reduce the bias but at the same time it increases overall noise influence on the IF and CPF estimate [14]. The IF estimation (Fig. 4a) for the VA-CPF-WD is denoted with 'win1-win2' where 'win1' is window used in the initial stage (CPF) while 'win2' is the window function used in the second stage (WD). It can be seen that the VA-WD [3] for this signal has some benefit from implementation of the smooth window (compare the lines corresponding to the $\mathrm{WD}_{\mathrm{Viterb}}$ and $\mathrm{WD}_{\mathrm{h\text{-}Viterb}}$). It has smaller bias due to the used window function while with increase of the noise influence the benefit of the smoothed window decreases and the WD with rectangular window becomes better for $SNR \leq$-6.5dB. Similar situation can be observed for the proposed VA-CPF-WD. The estimators with smoothed window in the CPF function has smaller MSE for $SNR > -6$dB while for the rectangular window it gives better results for higher noise $SNR \leq -6$dB. It is interesting to see that benefit can only be observed for the smoothed window usage for the CPF while the type of the window in the WD does not influence accuracy of the VA-CPF-WD estimator. The proposed estimator outperforms the VA-WD for $SNR \leq -4$dB. Since the primary goal of the VAs in the IF and CR estimation is work in extreme conditions, the proposed algorithm has justified its development.

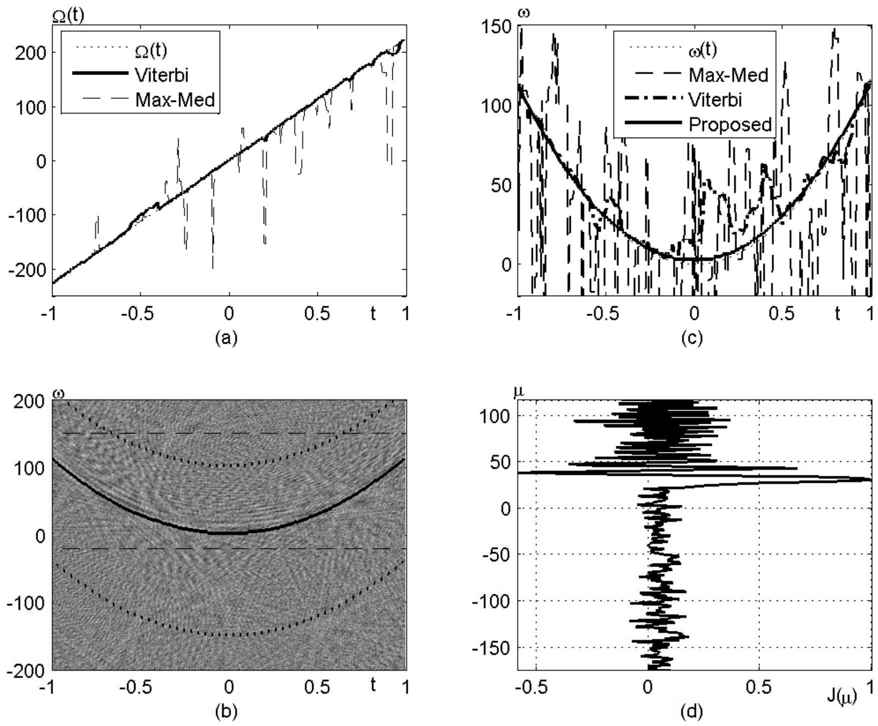

Fig. 2. Cubic phase signal corrupted with Gaussian noise with SNR=-7dB: (a) CR estimation; (b) WD with IF estimate (solid line), dotted lines represent parallel IFs bounding the search region; (c) IF estimation (region depicted with parallel dashed lines from (b) is enlarged here); (d) Function $J(\mu)$ used for IF estimation, its peak corresponds to the IF estimate denoted with solid line in Fig. 2b.

V. Conclusion and discussion

Novel form of the VA for the IF estimation, where in the initial stage the CR estimation is performed using the VA-CPF has been proposed. In the second stage, the IF estimate is performed using the WD and the CR estimate from the previous stage. The algorithm gives the same accuracy as the existing technique for the linear FM signal, it is significantly better for cubic-phase signals and has visible improvement for other non-linear modulations. The smooth windows in the CPF can be helpful for reducing the bias in the IF and CR estimation but for high noise environments it is better to use the rectangular window. Note that the study of VA parameters in the path penalty function is not performed. They are set according to our experience from the VA-WD and we believe that they are close to the optimal. It should be checked in depth in the further research. From our point of view, a more important issue is possibility for further improvement of the algorithm. Namely, the path penalty functions in the WD assume maximal expected change of the IF (parameter Δ). However, that change is proportional to the CR that can be estimated using the VA-CPF. In the presented realization, information propagates from the CR toward IF estimation but also information on the IF can be used in estimation of the CR, or joint estimation of the IF and CR can be performed using one set of path penalty functions. We again emphasize that the main purpose of the VA-based techniques is parameter estimation for high noise environments and all potential refinements should be concentrated to decreasing SNR under which a developed technique can produce accurate results.

The proposed technique is compared with sophisticated IF and CR estimation tools such

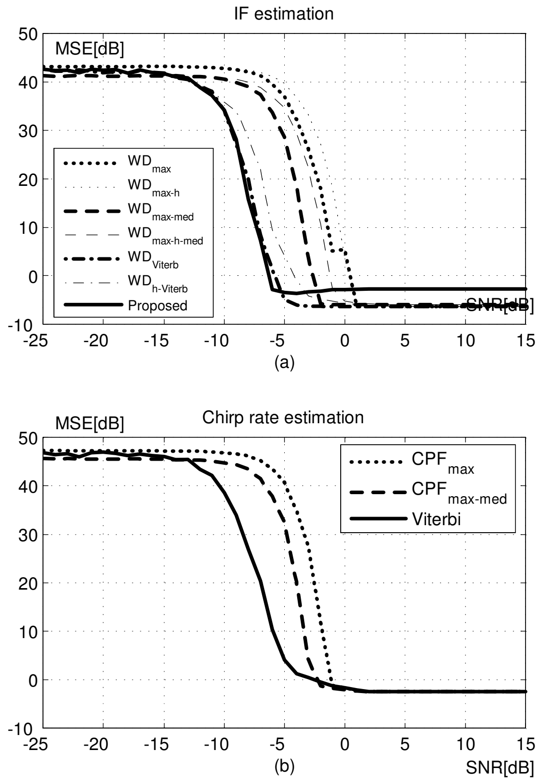

Fig. 3. Linear FM signal: (a) MSE for IF estimation; (b) MSE for CR estimation

as the HO-WD and the PHAF. It has been shown that the proposed technique significantly outperforms these alternatives for a low SNR environment. However, the VA, as a sort of universal tool for the non-parametric estimation of non-stationary signal features, can be applied to both the HO-WD and the PHAF and it could be considered in a future research.

VI. Acknowledgment

This research is supported in part by the Ministry of Science and Education of Montenegro.

References

[1] A. J. Viterbi, "Error bounds for convolutional codes and an asymptotically optimum decoding algorithm," *IEEE Transactions on Information Theory,* Vol. 13, No. 2, Apr. 1967, pp. 260-269.

[2] G. D. Forney, "The Viterbi algorithm," *Proceedings of the IEEE*, Vol. 61, No. 3, Mar. 1973, pp. 268-278.

[3] I. Djurović, LJ. Stanković, "An algorithm for the Wigner distribution based instantaneous frequency estimation in a high noise environment", *Signal Processing*, Vol. 84, No. 3, Mar. 2004, pp. 631-643.

[4] B. Boashash, "Estimating and interpreting the instantaneous frequency of a signalPart 1: Fundamentals," *Proc. IEEE*, Vol. 80, No. 4, Apr. 1992, pp. 520-538.

[5] B. Boashash, "Estimating and interpreting the instantaneous frequency of a signalPart 2: Algorithms and applications," *Proc. IEEE*, Vol. 80, No. 4, Apr. 1992, pp. 540-568.

[6] T. Thayaparan, LJ. Stanković, I. Djurović, "Micro-Doppler human signature detection and its application to gait recognition and indoor imaging", *Journal of the Franklin institute*, Vol. 345, No. 6, Sept. 2008, pp. 700-722.

[7] C. Cornu, I. Djurović, C. Ioana, A. Quinquis, LJ. Stanković, "Time-frequency detection using Gabor filter banks and Viterbi based grouping algorithm," in Proc. of *IEEE ICASSP'2005*, Philadelphija, USA, Mar. 2005, Vol. 4, pp.497-500.

[8] F. Hlawatsch, G. F. Boudreaux-Bartels, "Linear and quadratic time-frequency signal representations", *IEEE Signal Processing Magazine*, Apr. 1992, Vol. 9, No. 2, 21-67.

[9] P. O'Shea, "A new technique for instantaneous frequency rate estimation," *IEEE Signal Processing Letters*, Vol. 9, No. 8, pp. 251-252, Aug. 2002.

[10] I. Djurović, T. Thayaparan, LJ. Stanković,

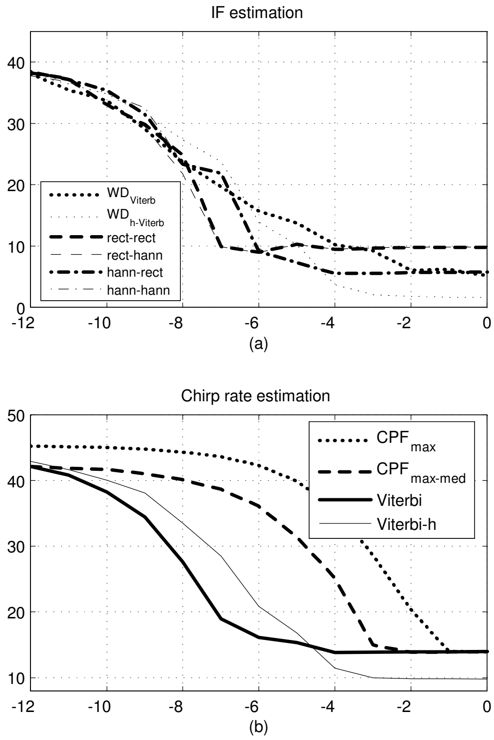

Fig. 4. Sinusoidally modulated signal: (a) MSE for IF estimation; (b) MSE for CR estimation

"Adaptive local polynomial Fourier transform in ISAR", *Journal of Applied Signal Processing*, Vol. 2006, Article ID 36093, 2006.

[11] I. Djurović, T. Thayaparan, LJ. Stanković, "SAR imaging of moving targets using polynomial FT", *IET Proceedings Signal Processing*, Vol. 2, No. 3, pp.237-246, 2008.

[12] LJ. Stanković, V. Katkovnik, "Instantaneous frequency estimation using higher order distributions with adaptive order and window length," *IEEE Trans. Information Theory*, Vol. 46, No. 1, Jan. 2000, pp.302-311.

[13] I. Djurović, C. Ioana, LJ. Stanković, P. Wang, "Adaptive algorithm for chirp-rate estimation", *EURASIP Journal on Advances in Signal Processing*, Vol. 2009, Article ID 727034, 9 pages, doi:10.1155/2009/727034.

[14] V. Katkovnik, LJ. Stanković, "Instantaneous frequency estimation using the Wigner distribution with varying and data driven window length", *IEEE Trans. on Signal Processing*, Vol. 46, No. 9, Sep. 1998, pp. 2315-2325.

[15] LJ. Stanković, "Local polynomial Wigner distribution," *Signal Processing*, Vol. 59, No. 1, Mar. 1997, pp. 123-128.

[16] S. Barbarossa, A. Scaglione, G. B. Giannakis, "Product higher-order ambiguity function for multicomponent polynomial-phase signal modeling," *IEEE Trans. Signal Processing*, Vol. 46, No. 3, Mar. 1998, pp. 691-708.

[17] A. Ouldali and M. Benidir, "Statistical analysis of polynomial phase signals affected by multiplicative and additive noise," *Signal Processing*, Vol. 78, No.1, Jan. 1999, 78, pp. 19-42.

[18] B. Porat, *Digital Processing of Random Signals, Theory and Methods*, Englewood Cliffs, NJ: Prentice-Hall, 1994