CPF-HAF estimator of polynomial-phase signals

Marko Simeunović and Igor Djurović

Abstract— **This letter introduces a simple and effective modification of the high-order ambiguity function (HAF). The number of phase differentiations (PD) is decreased for two with respect to the HAF. The obtained signal has cubic phase whose parameters can be estimated using the cubic phase function (CPF) in the final stage. The SNR threshold of the proposed modification is reduced for 9 dB with respect to the HAF, while the MSE is about 2 dB less.**

I. Introduction:

Polynomial-phase signal (PPS) model is widely used for modeling signals in radar, sonar, biomedicine, communications, etc. [1]. Numerous techniques for the parameter estimation of PPSs have been proposed in the last two decades. Several performance measures are used for these techniques: mean squared error (MSE) between estimated and exact parameters; signal-to-noise ratio (SNR) threshold; calculation complexity. The complexity and accuracy (low MSE and SNR threshold) are usually conflicting requirements. For example, the maximum-likelihood estimator is the most accurate technique, but it requires a multidimensional search. The HAF serves as an alternative [2]. It is a phase differentiation technique which decrements the order of polynomial in the signal phase in each stage of the procedure. The obtained signal in the final stage is a (complex) sinusoid that can be efficiently processed with FFT-algorithms and 1D search. However, the efficiency of the HAF procedure is followed by: higher SNR threshold, higher MSE, cross-terms in case of multicomponent signals, error propagation effect. The modification of the HAF, inspired by the recently proposed CPF [3], is presented in this letter. It improves the accuracy of the HAF-based technique without increasing the calculation complexity.

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II. HAF

Consider the following signal model:

$$y(n) = x(n) + v(n),\ n \in [-N/2, N/2],$$

$$x(n) = Ae^{j\phi(n)} = Ae^{j\sum_{i=0}^{P} a_i n^i}, \quad (1)$$

where $x(n)$ is a P-order PPS, $v(n)$ complex zero-mean white Gaussian noise with variance σ^2, A the amplitude and $\phi(n)$ the polynomial phase with parameters $\{a_i,\ i{=}0,\ 1,\ \ldots,\ P\}$. Here, the goal is to estimate the parameters of $x(n)$ from $y(n)$. The PD operator is defined recursively as

$$PD^1[n,\tau] = y(n+\tau)y^*(n-\tau)$$
$$\ldots$$
$$PD^P[n,\tau] =$$
$$= PD^{P-1}[n+\tau,\tau]\{PD^{P-1}[n-\tau,\tau]\}^*, \quad (2)$$

where τ is the lag coefficient. In each stage, the PD operator decreases the phase polynomial order by one. Since $PD^{P-1}[n,\tau]$ is a complex sinusoid with frequency proportional to the highest order phase coefficient [4,5],

$$f = 2^{P-1}P!a_P\tau^{P-1}, \quad (3)$$

a_P can be estimated by maximizing the absolute value of function

$$HAF^P(f) =$$
$$= \sum_{n=-N/2+(P-1)\tau}^{N/2-(P-1)\tau} PD^{P-1}[n,\tau]e^{-jfn}. \quad (4)$$

Function (4) is known as the HAF. When a_P is obtained, phase coefficients a_{P-1} can be estimated with the same procedure applied on the dechirped signal $y_d(n) = y(n)\exp(-ja_P n^P)$.

The procedure is repeated until all the coefficitients are estimated. The HAF-based parameter estimation is an efficient procedure requiring the FFT calculation and 1D search for parameter. However, $PD^Q[n,\text{t}]$ has 2t samples less than $PD^{Q-1}[n,\text{t}]$. In addition, PD^P calculation requires the multiplication of 2^P signal terms causing reduced accuracy for moderate noise power and numerous cross-terms when multicomponent signals are considered. Each PD calculation increases the SNR threshold for 6dB [4]. Also, dechirping procedure causes error propagation from higher to lower order phase coefficients. In the sequel, we propose a modification of the HAF with significantly improved accuracy.

III. Modification of the HAF

Instead of performing P-1 PD operations, we consider P-3 PDs of (1) producing

$$\begin{aligned} PD^{P-3}[n,\tau] = A^{2^{P-3}} \exp(j(C_3^P a_P n^3 \\ +C_2^P a_{P-1} n^2 + (C_{11}^P a_P + C_{12}^P a_{P-2})n \\ +C_{01}^P a_{P-1} + C_{02}^P a_{P-3})) + x_v(n), \end{aligned} \tag{5}$$

where $x_v(n)$ is the noise term and coefficients ${C_3}^P$, ${C_2}^P$, ${C_{11}}^P$, ${C_{12}}^P$, ${C_{01}}^P$, ${C_{02}}^P$ depend only on P and τ. For example, coefficients ${C_3}^P$ and ${C_2}^P$ are:

$$\begin{aligned} C_3^P &= \frac{2^{P-3}\tau^{P-3}P!}{3!} \\ C_2^P &= \frac{2^{P-3}\tau^{P-3}(P-1)!}{2!}, \quad P \geq 4. \end{aligned} \tag{6}$$

Signal (5) has cubic phase whose two highest order coefficients can be estimated using the CPF [3]. First, two CPFs are evaluated as

$$MHAF_0(\Omega) = \sum_{p=-N/2+(P-3)\tau}^{N/2-(P-3)\tau} PD^{P-3}[p,\tau] PD^{P-3}[-p,\tau] e^{-j\Omega p^2}, \tag{7}$$

$$MHAF_1(\Omega) = \sum_{p=-N/2+|n_1|+(P-3)\tau}^{N/2-|n_1|-(P-3)\tau} PD^{P-3}[n_1+p,\tau] \times PD^{P-3}[n_1-p,\tau] e^{-j\Omega p^2}, \tag{8}$$

where $n_1 = \lfloor 0.11(N-1-2(P-3)\tau) \rfloor$ ($\lfloor \rfloor$ denotes the integer part) is set according to [3]. The next step is to locate the positions of the CPFs maxima, i.e.

$$\begin{aligned} \Omega_0 = \arg\max_{\Omega} |MHAF_0(\Omega)| \\ \Omega_1 = \arg\max_{\Omega} |MHAF_1(\Omega)|. \end{aligned} \tag{9}$$

Now the parameters a_P and a_{P-1} can be estimated from W_0 and W_1 as

$$\begin{aligned} \hat{a}_{P-1} = \frac{\Omega_0}{(P-1)!2^{P-3}\tau^{P-3}} \\ \hat{a}_P = \frac{\Omega_1 - \Omega_0}{P!2^{P-3}\tau^{P-3}n_1}. \end{aligned} \tag{10}$$

In this manner, the two highest order signal coefficients are estimated at once as opposed to the standard HAF-based procedure where each coefficient is estimated in separated stage of the procedure. Furthermore, coefficients a_{P-2} and a_{P-3} can be estimated from dechirped signal $PD^{P-3}[n,\tau]\exp(-j(C_3\hat{a}_P n^3 + C_2\hat{a}_{P-1}n^2 + C_{11}^P\hat{a}_P n + C_{01}^P\hat{a}_{P-1}))$, while the remaining coefficients can be estimated from $y(n)\exp(-j(\hat{a}_P n^3 + \hat{a}_{P-1}n^2 + \hat{a}_{P-2}n + \hat{a}_{P-3}))$. Therefore, in each stage of the proposed procedure we are able to estimate at least two phase coefficients and to significantly reduce the error propagation effect to lower order coefficients. In addition, (7) and (8) have more non-zero terms than the HAF (4). Moreover, $PD^{P-3}[p,\tau]PD^{P-3}[-p,\tau]$ represents the multiplication of 2^{P-2} terms implying less cross-terms in the multicomponent signals case than with the HAF. Therefore, the accuracy improvement (lower SNR threshold and MSE), reduced error-propagation effect (more coefficients are estimated at once) and smaller number of cross-terms for multicomponent signals are archived without significant increase of complexity. It will be shown below that the proposed modification produces significantly better results than the standard HAF-based technique. We will refer to the proposed technique as the CPF-HAF.

IV. Numerical example

To evaluate the proposed modification, we estimated the parameters of the following two

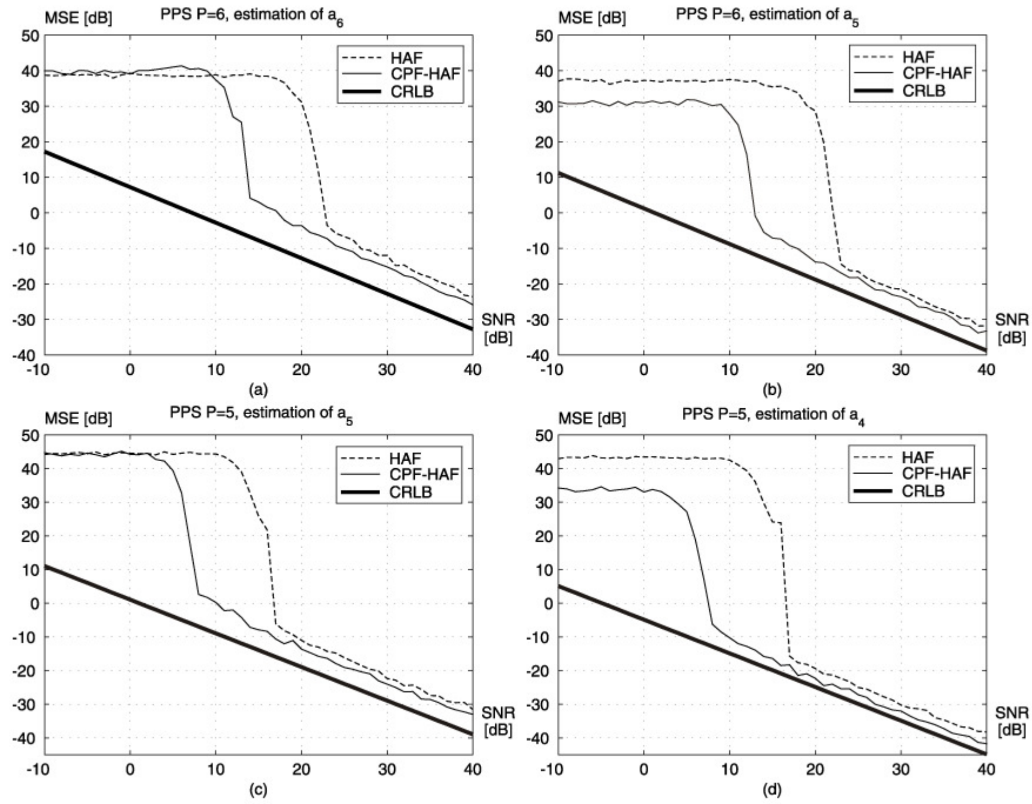

Fig. 1. MSEs of a_P and a_{P-1} (P=5 and P=6) coefficients estimated by the HAF and CPF-HAF:(a) MSEs of a_6 coefficient for the sixth order PPS, (b) MSEs of a_5 coefficient for the sixth order PPS, (c) MSEs of a_5 coefficient for the fifth order PPS, and (d) MSEs of a_4 coefficient for the fifth order PPS.

PPSs:

$$x_1(t) = e^{j\pi(12t+23t^2+13t^3+8t^4+6t^5)},$$
$$x_2(t) = e^{j\pi(12t+23t^2+13t^3+8t^4+6t^5+3t^6)},$$

by the HAF and CPF-HAF. Total number of signal samples is 257 and t is in interval [-1,1]. Lag $\tau = \lfloor (N+1)/2P \rfloor$ is chosen according to the instructions from [5]. The MSEs of a_P and a_{P-1} estimates, obtained by the Monte Carlo simulations with 200 trials, are shown in Fig. 1.

The CPF-HAF has about 9dB lower SNR threshold than the HAF: the SNR threshold of the HAF is 17 dB for P=5 and 23 dB for P=6, while the CPF-HAF has thresholds 8 dB for P=5 and about 14 dB for P=6. The MSEs of the CPF-HAF above the thresholds are 2-3 dB lower than the MSEs of the HAF for all the considered phase coefficients.

V. Conclusion

The hybrid CPF-HAF estimator is proposed. It reduces the number of PD operations for two with respect to the HAF-based technique. The SNR threshold and MSE of the proposed algorithm are significantly lower than with the standard HAF approach. In addition, the error propagation effect is decreased since in each stage of the procedure several phase coefficents is estimated. Finally, the complexity of the algorithm is not increased significantly with respect to the HAF-based technique.

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