# Robust $M$-periodogram with dichotomous search

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*Abstract*— **The problem of sinusoidal frequency estimation in heavy-tailed noise environment is addressed. A method based on the robust $M$-periodogram is proposed. Specifically, a suboptimal coarse frequency estimate provided by the robust $M$-periodogram is improved using the modified dichotomous search. Simulations that consider most common heavy-tailed noise models demonstrate that the proposed method outperforms several recently proposed methods. The method can be readily extended to deal with multiple sinusoids.**

## I. Introduction

Frequency estimation of sinusoidal signals corrupted by additive Gaussian noise is often encountered in practice. The topic is relevant to numerous applications, including radar, sonar and speech processing, to name a few. It has been extensively dealt with in the literature in the past several decades [1–8]. The maximum likelihood (ML) estimate of the frequency is given by the location of the periodogram's highest peak [1]. This approach can be effectively implemented using the fast Fourier transform (FFT) algorithm. The periodogram maximization is usually performed in two steps, coarse search and fine search. Coarse search represents finding the maximum bin of the FFT of a noisy sinusoid. Fine search represents refining the coarse estimate through the interpolation or some iterative method. Numerous iterative maximizations have been proposed in the literature. In [3] and [4], estimators that interpolate the true signal frequency using two discrete Fourier transform (DFT) coefficients from either side of the maximum bin are proposed. These algorithms, however, have frequency-dependent performance that is worst when the frequency displacement $\delta$ of the true signal frequency from a DFT bin is zero. In [5], an iterative binary search is proposed, a method referred to as the dichotomous search of periodogram peak. It is necessary, however, to zero-pad the data prior to the coarse search. In [7], a modified version of the dichotomous search that achieves the CRB without the zero-padding is presented. In [8], two asymptotically unbiased fine search iterative procedures are proposed; their estimation variance exceeds the CRB for about 1.5% for Gaussian noise environment.

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In many important applications (e.g., radar, HF and cellular communication, underwater acoustics), the additive noise exhibits heavy-tailed nature. Unfortunately, the standard DFT-based techniques fail to produce satisfactory results in such environments. Methods based on the robust statistics [9] appeared as a solution to this problem. Of particular importance is the robust $M$-periodogram [10], [11], where a nonquadratic loss function is used for fitting of observations corrupted by noise with unknown heavy-tailed distribution. Specifically, the absolute value loss function offers a radical improvement of the periodogram quality in terms of the resolvability of signal peaks. In [12], the marginal-median DFT is proposed as an alternative to the standard DFT. The main drawback to this method is the spectral distortion [12] due to the fact that each marginal-median DFT sample equals one modulated signal sample (odd signal length) or the average of two modulated samples (even signal length). The spectral distortion can be reduced using the L-filter DFT (L-DFT) forms [13]. A robust frequency estimation method based on the L-DFT is proposed in [14]. In a non-Gaussian noise environment, the amplitude and frequency of a sinusoid can be estimated more accurately than the Gaussian CRB suggests [15]. The ML estimator derived under the condition of Laplace white noise is able to attain an asymptotic CRB that is one half of that achieved by periodogram maximization and nonlinear least squares [15].

In this paper, we propose a method for robust sinusoidal frequency estimation. In the robust $M$-periodogram, the maximum bin's position represents the coarse estimate, which is refined using a modified dichotomous search. The modified dichotomous search does not require zero-padding of the data, which is a very desirable property since the robust $M$-periodogram is calculated iteratively at each frequency. The power of FFT algorithms cannot be fully harnessed here.

The paper is organized as follows. A brief overview of the robust $M$-periodogram is given in Section 2. The proposed frequency estimation refinement is presented in Section 3. Simulations are given in Section 4 and conclusions are drawn in Section 5.

## II. Robust $M$-periodogram

Consider a constant amplitude complex sinusoid embedded in a heavy-tailed noise $\nu(n)$,

$$y(n) = Ae^{j(\omega_0 nT+\phi)} + \nu(n), \quad n = 0, \cdots, N-1, \tag{1}$$

where $A$, $\omega_0$ and $\phi$ are unknown real-valued amplitude, frequency and phase, respectively, $T$ the sampling interval and $N$ the number of samples. Our goal is to estimate $\omega_0$.

The $M$-estimates $\hat{\omega}$ and $\hat{C}$ of the frequency and amplitude, respectively, are introduced as a solution to the following optimization problem [11]:

$$(\hat{\omega}, \hat{C}) = \underset{C, \omega \in Q_\omega}{\arg\min}\ J(\omega, C), \tag{2}$$

where

$$J(\omega, C) = \sum_n \rho(n) \left(F(e_R(n)) + F(e_I(n))\right), \tag{3}$$

$$e(n) = y(n) - Ce^{j\omega nT},$$
$$e_R(n) = \mathrm{Re}(e(n)), \qquad e_I(n) = \mathrm{Im}(e(n)),$$
$$Q_\omega = \{\omega | -\frac{\pi}{T} < \omega < \frac{\pi}{T}, \omega \neq 0\}.$$

In (3), $\rho(n)$ is a non-negative window function and $F(x)$ is a convex non-negative loss function.

In particular, when $F(x) = x^2$, definition (2) yields the standard periodogram [10], [11]. In this paper, we will consider the absolute value loss function, $F(x) = |x|$, since it is asymptotically optimal in the minimax sense for two very important classes of noise distributions [11],

- class of nonsingular distributions, i.e., when nothing is known about the noise distribution except that its p.d.f. $g(x)$ satisfies $g(0) > 0$, and
- class of approximate exponential distributions,

$$g(x) = (1-\gamma) f_0(x) + \gamma f_1(x), \quad 0 < \gamma < 1,$$

where $f_0(x)$ is the Laplace distribution and $f_1(x)$ is an arbitrary distribution.

The robust $M$-periodogram is defined as a function [11]

$$I_{\mathrm{R}}(\omega) = J(0,0) - J(\omega, C(\omega)), \tag{4}$$

where

$$J(0,0) = \sum_n \rho(n) \left(F(\mathrm{Re}(y(n))) + F(\mathrm{Im}(y(n)))\right),$$

and $C(\omega)$ is a minimizer of $J(\omega, C)$ provided a fixed value of $\omega$, i.e.

$$C(\omega) = \underset{C}{\arg\min}\ J(\omega, C).$$

The function $I_{\mathrm{R}}(\omega)$ is calculated for each $\omega \in Q_\omega$ and the frequency is estimated as

$$\hat{\omega} = \underset{\omega \in Q_\omega}{\arg\max}\ I_{\mathrm{R}}(\omega). \tag{5}$$

In practice, the set $Q_\omega$ is given by a grid of Fourier frequencies

$$Q_\omega = \{\omega | \omega_k = \frac{\pi}{NT} k, k = -N+1, \cdots, N-1\}. \tag{6}$$

Let us denote the frequency estimation error with $\Delta\omega$ and distribution function of noise $\nu(n)$ as $G(x)$. From (13) and (14) in [11], we see that the robust $M$-periodogram is the unbiased estimator of the complex sinusoid frequency with the asymptotic variance of

$$\mathrm{Var}[\Delta\omega] = V(F, G) \frac{T}{A^2 h^3} W_\omega + o(T/h^3), \tag{7}$$

where

$$V(F,G) = \frac{\int_{-\infty}^{\infty} (F^{(1)}(x))^2 \mathrm{d}G(x)}{\left(\int_{-\infty}^{\infty} F^{(2)}(x)\mathrm{d}G(x)\right)^2}, \tag{8}$$

$h$ is the window length, $W_\omega$ is a window-dependent parameter and $\lim_{x\to\infty} o(x)/x = 0$. The term $V(F,G)$ completely and solely describes the influence of the noise distribution and cost function on the variance.

## III. Dichotomous search of the robust $M$-periodogram peak

In this section, we present a robust $M$-periodogram maximization method based on the dichotomous search proposed in [5]. The dichotomous search is a binary search method, where we first locate the DFT peak, then take two DFT coefficients from either side of the peak and adjust the frequency estimation toward the larger coefficient. We calculate new DFT coefficient halfway between the peak and the larger coefficient. The position of the calculated coefficient represents improved frequency estimation over the initial one. The procedure is iterated $Q$ times. A drawback to this method is the need to zero-pad the data to a length of at least $1.5N$, in order to approach the CRB.

In [7], a modification to the dichotomous search that attains the CRB without the zero-padding is proposed. Here we will use the same approach with slightly different initialization step. It is given below.

*Step 1.* Calculate the robust periodogram $I_\mathrm{R}(\omega)$ at the grid of Fourier frequencies and find position $\omega_m$ of the maximum of $I_\mathrm{R}(\omega)$. Denote $I_0 = I_\mathrm{R}(\omega_m)$. Set $\Delta\omega = \frac{\Delta\omega}{2}$, where $\Delta\omega$ is the frequency resolution, and calculate $I_\mathrm{R}(\omega)$ at two points $\omega_m \pm \Delta\omega$, i.e.

$$I_{\pm 1} = I_\mathrm{R}(\omega_m \pm \Delta\omega),$$

*Step 2.* Iterate $Q$ times

$\Delta\omega = \dfrac{\Delta\omega}{2}$
if $I_1 > I_{-1}$ then
$\quad I_{-1} = I_0$ and $\omega_m = \omega_m + \Delta\omega$
else
$\quad I_1 = I_0$ and $\omega_m = \omega_m - \Delta\omega$
calculate $I_\mathrm{R}(\omega_m)$ and set $I_0 = I_\mathrm{R}(\omega_m)$,

The final frequency estimation is $\hat{\omega} = \omega_m$.

In each iteration, the frequency resolution $\Delta\omega$ is halved, i.e., improved two times. It can be shown that the final frequency resolution, after $Q$ iterations, equals

$$\Delta\omega = \frac{2\pi}{TN2^{Q+1}}. \tag{9}$$

The frequency resolution should be small enough to allow for accurate estimation. One approach to defining the number $Q$ is to reduce $\Delta\omega$ until it is smaller than the square root of the CRB [6]. However, in [6], the frequency estimation in Gaussian environment is considered with the CRB [1]

$$\sigma_\omega^{CR} = \sqrt{\frac{6\sigma^2}{T^2N(N^2-1)A^2}}. \tag{10}$$

In the robust $M$-periodogram case, instead of the CRB, we can consider the asymptotic variance, which could be easily obtained from (10) by substituting the noise variance $\sigma^2$ with $V(F,G)$ defined in (8). This is a consequence of the fact that the influence of the noise distribution and cost function are fully incorporated within $V(F,G)$. Therefore, the criterion for defining $Q$ could be

$$\Delta\omega < \sqrt{\frac{6V(F,G)}{T^2N(N^2-1)A^2}} \quad \Rightarrow$$
$$Q < \log_2\left(\pi A\sqrt{\frac{N^2-1}{6NV(F,G)}}\right). \tag{11}$$

In [8], the frequency displacement is calculated iteratively; in each iteration, the displacement $\delta$ is updated by a value that depends on ratio $(X_{0.5}+X_{-0.5})/(X_{0.5}-X_{-0.5})$, where $X_{-0.5}$ and $X_{0.5}$ are DFT bins displaced by $-0.5$ and $0.5$ from the current frequency estimate, respectively. This approach, however, is optimal for the Gaussian noise environment. We find the dichotomous search a more natural choice, since the only assumption regarding the peak shape is that it is a monotonically increasing function in the interval $[\omega_t - \frac{\Delta\omega}{2}, \omega_t]$ and monotonically decreasing in $[\omega_t, \omega_t + \frac{\Delta\omega}{2}]$, where $\omega_t$ is the true signal frequency.

The proposed method can be readily extended to the multiple sinusoids case provided that sinusoids are well separated in frequency so that mutual influence of components can be neglected. The standard periodogram-based techniques cannot resolve sinusoidal frequencies that differ by less than one cycle per unit time. In that case, robust high-resolution spectral methods should be used [16].

Frequency estimation of multiple sinusoids can also be performed using the coarse and fine search strategy. If we assume that the number of components, $K$, is known, the coarse search is performed by locating $K$ strongest spectral peaks. The fine search is performed for each located peak separately.

## IV. Simulations

In the experiment, we consider a complex sinusoid

$$s(n) = e^{j((\omega_0+\delta)nT+\phi)}, \quad n = 0, 1, \cdots, N-1, \tag{12}$$

where $\omega_0 = 2\pi k_0/T$ is a DFT frequency closest to the true sinusoid frequency $\omega_0 + \delta$, $\delta$ is the frequency displacement from the grid and $k_0$ is an integer from interval $[0, N-1]$. Herein, we will set $T = 1$ and $k_0 = 12$, and, in each trial of the Monte Carlo simulations, both $\delta$ and $\phi$ will be selected randomly with uniform distribution, $\delta$ on interval $[-\pi/(NT), \pi/(NT)]$ and $\phi$ on $[-\pi, \pi]$.

As for the additive random noise $\nu(n)$, we assume complex model

$$\nu(n) = \nu_R(n) + j\nu_I(n), \tag{13}$$

where real and imaginary parts $\nu_R(n)$ and $\nu_I(n)$ are i.i.d. variables. We consider four noise models,

- zero-mean Gaussian noise with variance $b/2$. The term $V(F,G)$ for $F(x) = |x|$ equals $V(F,G) = \pi b/4$.
- Cauchy noise with the p.d.f. $g(x) = \gamma/[\pi(x^2 + \gamma^2)]$. For $F(x) = |x|$, we have $V(F,G) = \pi^2\gamma^2/4$.
- Laplace noise with the p.d.f. $g(x) = e^{-|x|/b}/(2b)$. For $F(x) = |x|$, we have $V(F,G) = b^2$.
- symmetric $\alpha$-stable noise (skewness $\beta = 0$) with the characteristic exponent $\alpha = 0.5$ and location parameter $\delta = 0$. The characteristic function of such a noise is $\varphi(t; 0.5, 0, \gamma, 0) = e^{-|\gamma t|^{0.5}}$. Since the p.d.f. of this noise cannot be determined analytically, we will calculate $V(F,G)$ numerically.

We compared the proposed method with the standard periodogram, robust $M$-periodogram without a fine search [11], marginal-median DFT method [12] and optimal L-DFT method [14]. Comparison is made for $N = 1024$ and is quantified in terms of the mean squared error (MSE) defined as

$$\text{MSE} = 10\log_{10}\frac{\sum_{k=1}^{N_{sim}}\left[\hat{\omega}_k - (\omega_0 + \delta_k)\right]^2}{N_{sim}}, \tag{14}$$

where $\hat{\omega}_k$ and $\delta_k$ are the estimated frequency and displacement in the $k$th simulation and $N_{sim}$ is the number of Monte Carlo simulations. In our simulations, $N_{sim} = 500$. In the proposed method, $Q = 10$ was used; the same number of iterations was used in the fine search in the marginal-median DFT and optimal L-DFT methods, where the iterative procedure from [8] has been adopted. The standard periodogram is maximized also using the dichotomous search approach with $Q = 10$.

The obtained MSE curves versus variable noise parameter are given in Fig. 1, where the upper left corner corresponds to the Gaussian noise, upper right to the Cauchy noise, lower left to the Laplace noise and lower right to the alpha-stable noise. In all the plots, the dotted line corresponds to the standard periodogram, solid line to the robust $M$-periodogram, thick dashed line to the proposed method, dash-dot line with triangles to the marginal-median DFT and dashdot line to the optimal L-DFT method. In addition, asterisks and squares

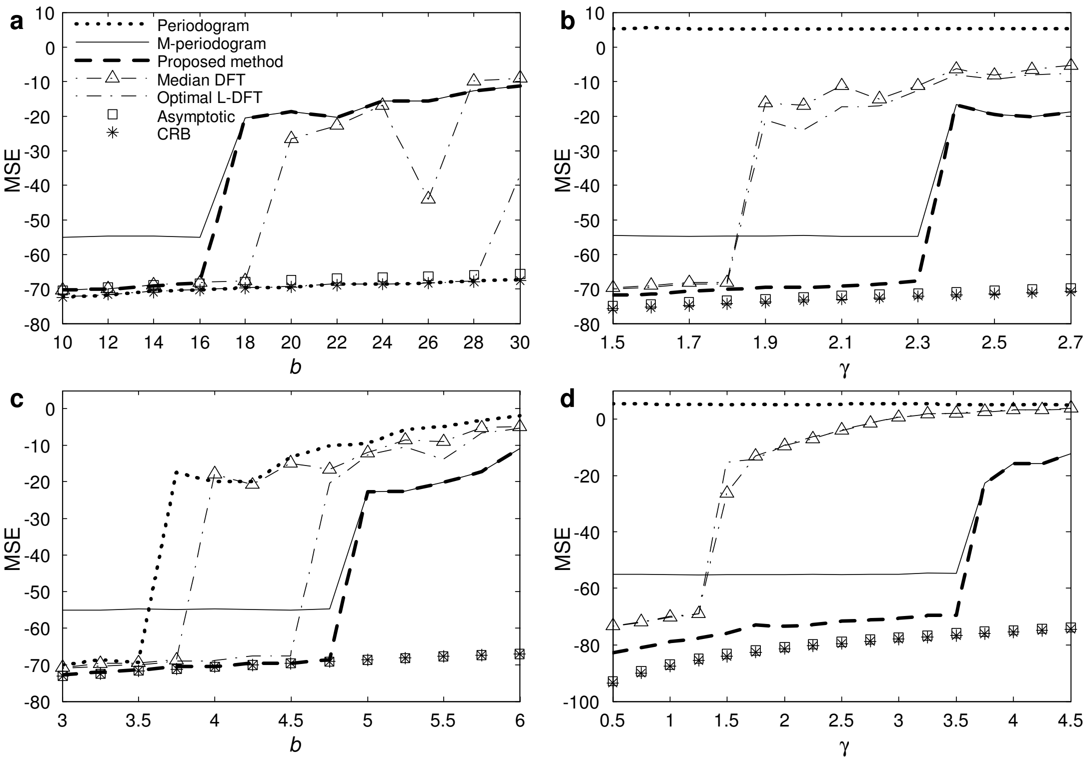


Fig. 1. MSE versus noise parameter. Considered noise models are (a) Gaussian noise, (b) Cauchy noise, (c) Laplacian noise, and (d) $\alpha$-stable noise. The legend in subplot (a) applies to all the subplots.

correspond to the CRB and asymptotic variance, respectively. The CRB is obtained from (7) and (8) with $F(x) = -\log(G(x))$ according to the ML principle. On the other hand, the asymptotic variance is obtained with $F(x) = |x|$, and, in case of Laplace noise, it equals the CRB. Otherwise, it exceeds the CRB.

As expected, in the Gaussian environment, the standard periodogram performs the best, whereas in the heavy-tailed environment it performs poorly. Specifically, it fails completely for each considered parameter of the Cauchy and $\alpha$-stable noises.

In comparison to the robust $M$-periodogram without a fine search, the proposed method shows the improved accuracy provided by the dichotomous fine search algorithm. The thresholds coincide since, in our method, the frequency estimate provided by the robust $M$-periodogram represents the coarse estimate.

As for the other methods, the proposed one exhibits superior performance in terms of accuracy and threshold for all the considered heavy-tailed noise types. The difference in performance is significant for the $\alpha$-stable noise when the threshold of the proposed method for the $\alpha$-stable noise is around $\gamma = 3.5$, which exceeds that of the methods based on the marginal-median DFT and optimal L-DFT about three times.

Although we used the preset value of $Q = 10$ in our simulations, the first three iterations in the algorithm provided the accuracy gain of around 6 dB per iteration and the following couple of iterations provided the rest. For each considered noise, performing the algorithm for $Q > 6$ provides no more gain than 0.1 dB.

We have also evaluated the proposed method in the frequency estimation of a multicomponent signal that contains three sinusoids with frequencies $\omega_1 = 11.179\frac{2\pi}{T}$, $\omega_2 = 25.431\frac{2\pi}{T}$ and $\omega_3 = 59.687\frac{2\pi}{T}$, and amplitudes $A_1 = 1.3$, $A_2 = 1$ and $A_3 = 0.8$. We considered the symmetric $\alpha$-stable noise with $\beta = 0$,

TABLE I
MSE VALUES IN THE FREQUENCY ESTIMATION OF THREE SINUSOIDS IN $\alpha$-STABLE NOISE

| | | Periodogram | $M$-periodogram | Proposed | Median DFT | Opt. L-DFT |
|---|---|---|---|---|---|---|
| $\gamma = 0.5$ | $\omega_1$ | 5.64 dB | -59.19 dB | -74.77 dB | -72.46 dB | -72.48 dB |
| | $\omega_2$ | 2.38 dB | -51.49 dB | -71.82 dB | -70.9 dB | -71.3 dB |
| | $\omega_3$ | 4.66 dB | -54.33 dB | -71.17 dB | -67.42 dB | -67.62 dB |
| $\gamma = 1.5$ | $\omega_1$ | 6.09 dB | -59.19 dB | -72.82 dB | -2.96 dB | -3.19 dB |
| | $\omega_2$ | 3.4 dB | -51.21 dB | -67.83 dB | -23.59 dB | -23.1 dB |
| | $\omega_3$ | 4.71 dB | -54.01 dB | -66.54 dB | -4.71 dB | -3.71 dB |

TABLE II
NUMBER OF OPERATIONS OF THE CONSIDERED ROBUST METHODS

| | Proposed | Median DFT | Opt. L-DFT |
|---|---|---|---|
| Additions | $2N(6+5K)(N+Q+2)$ | $2N(N+2Q)+2Q$ | $N(5N+\frac{3}{2}N_\alpha+6Q)+2Q$ |
| Multiplications | $N(16+9K)(N+Q+2)$ | $4N(N+2Q)$ | $N(8N+N_\alpha+8Q)$ |
| Divisions | $(N+1)(K+1)(N+Q+2)$ | $Q$ | $2N+2N_\alpha+5Q$ |
| Sines/Cosines | $2N(Q+2)$ | $4QN$ | $4QN$ |
| Sorts | — | $2N+4Q$ | $4(N+Q)$ |

$\alpha = 0.5$, $\delta = 0$ and two values of $\gamma$, $\gamma = 0.5$ and $\gamma = 1.5$. The MSE values, obtained after 100 trials, are given in Table I. The methods based on the marginal-median DFT and optimal L-DFT completely fail for $\gamma = 1.5$. Due to the presence of multiple sinusoids, the MSE for each sinusoid is bit higher than in the single sinusoid case.

Finally, the computational complexity of the considered robust methods is given in Table II. All operations are real. In the proposed method, $K$ corresponds to the number of iterations of the optimization algorithm (usually $3-5$, [11]). In the optimal L-DFT method, $N_\alpha$ is the number of considered $\alpha$s in the optimization procedure. In addition, the average number of additions is shown for this method. In the last row, one sort corresponds to the sorting of an $N$-element real array. Recall that the average complexity of the quick sort is $\mathrm{O}(N\log(N))$, where O represents the big-O notation. From Table II, we see that the complexity of the proposed method, $\mathrm{O}(KN^2)$, is primarily due to the arithmetic operations, whereas the complexity of the other two methods, $\mathrm{O}(N^2\log(N))$, is primarily due to sortings. Knowing that $K$ is relatively small, we can conclude that all the three methods have approximately the same complexity.

## V. Conclusion

In this paper, we proposed a method for sinusoidal frequency estimation in heavy-tailed noise environment. The method is based on the robust $M$-periodogram, i.e., it takes a suboptimal frequency estimate obtained by the robust $M$-periodogram as the coarse estimate and refines it using the dichotomous search. The proposed method is compared to several recently proposed methods in the field. For the considered heavy-tailed noise types, namely the Cauchy noise, Laplace noise and $\alpha$-stable noise, it outperforms other methods in the frequency estimation of both single and multiple sinusoids.

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