Aliasing detection and resolving in the estimation of polynomial-phase signal parameters

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Abstract— **A novel method for aliasing detection and resolving in the estimation of polynomial-phase signal (PPS) parameters is presented. Aliasing is detected using two high-order ambiguity functions (HAFs) of a uniformly sampled PPS embedded in noise. If aliasing occurred, we propose a way of recovering the true parameters from their aliased positions. To that end, a closed-form expression for the true parameter value is derived. As opposed to the concurrent methods, the proposed method provides much more robust results with higher order PPSs and does not require nonuniform sampling. In addition, it can be readily extended to the multicomponent PPS case. Simulations support the theoretical results.**

I. Introduction

Polynomial-phase signals (PPSs) are found in numerous application fields including radar, sonar, seismology, biomedicine and radio communication. Therefore, a significant attention has been paid to the estimation of PPS parameters [1–6]. A particularly popular approach entails finding the high-order instantaneous moment (HIM) of the considered PPS [2–5]. When the order of the HIM and that of the PPS coincide, the HIM outputs a complex sinusoid whose frequency is proportional to the highest order phase coefficient. The sinusoid frequency is then estimated from the samples of the discrete Fourier transform (DFT) of the HIM, which can be done using well developed sinusoid frequency estimation techniques [7–9]. The DFT of the HIM is referred to as the high-order ambiguity function (HAF), originally known as the polynomial-phase transform (PPT) [2].

Despite the significant interest in PPSs, aliasing of polynomial-phase parameters has not been addressed to any great degree. The parameter is said to be aliased if the frequency of the sinusoid output by the HIM violates the Nyquist criterion. The sinusoid frequency depends on the value of lags used in the HIM calculation. In some applications, however, this issue is of crucial importance. In direct sequence spread-spectrum (DS-SS) systems, for example, an adversary can transmit a highly nonstationary jammer whose phase cannot be approximated by a polynomial with non-aliased parameters within the considered time interval [10]. As a result, the jammer cannot be properly modeled and suppressed. In general, aliasing can occur in any application where the underlying signal is undersampled.

In [11], the authors presented a way to recover the true PPS parameters from their aliased positions by using two coprime lags and solving linear Diophantine equations. The dynamic parameter range obtained in [11] is the maximal one for monocomponent PPSs [12]. In [13], the author showed that aliasing can be avoided by a nonuniform sampling, i.e., by adopting an irrational interval between some of the samples.

In this paper, we show that the method proposed in [11] works only with low PPS order. We present a way to detect aliasing from maxima positions of two HAFs of a uniformly sampled PPS embedded in noise. If aliasing occurred, we propose how to recover the true parameters without solving Diophantine equations.

Paper is organized as follows. Section 2 covers the HAF-based PPS parameter estimation, the problem of aliasing and one way to resolve it [11]. The proposed method is presented in Section 3. Simulations are presented in Sec-

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tion 4, and conclusions are drawn in Section 5.

II. HAF-based PPS parameter estimation

Consider a uniformly sampled PPS

$$x(n) = Ae^{j2\pi \sum_{m=0}^{P} \alpha_m (n\Delta)^m}, n = 0, \cdots, N-1, \quad (1)$$

where A is the amplitude, α_m the polynomial coefficients, Δ the sampling interval and N the number of samples. The multilag HIM (ml-HIM) of $x(n)$ is defined as [5]

$$\begin{aligned} x_1(n) &= x(n) \\ x_2(n;\boldsymbol{\tau}_1) &= x_1(n+\tau_1)x_1^*(n-\tau_1), \\ x_3(n;\boldsymbol{\tau}_2) &= x_2(n+\tau_2;\boldsymbol{\tau}_1)x_2^*(n-\tau_2;\boldsymbol{\tau}_1), \\ &\vdots \\ x_P(n;\boldsymbol{\tau}_{P-1}) &= x_{P-1}(n+\tau_{P-1};\boldsymbol{\tau}_{P-2}) \\ &\quad \times x_{P-1}^*(n-\tau_{P-1};\boldsymbol{\tau}_{P-2}), \end{aligned} \quad (2)$$

where $\boldsymbol{\tau}_i = [\tau_1, \tau_2, \cdots, \tau_i]$, $i = 1, ..., P-1$, are sets of used time lags. In $x_k(n;\boldsymbol{\tau}_{k-1})$, $k = 1, 2, \cdots, P$, index n goes from $\sum_{i=1}^{k-1} \tau_i$ to $N - \sum_{i=1}^{k-1} \tau_i - 1$. The multilag HAF (ml-HAF) is defined as the DFT of the ml-HIM,

$$X_P(f;\boldsymbol{\tau}_{P-1}) = \sum_{n=0}^{N-2\sum_{k=1}^{P-1}\tau_k - 1} x_P(n;\boldsymbol{\tau}_{P-1}) e^{-j2\pi f n}. \quad (3)$$

When $x(n)$ is a Pth order PPS, $x_P(n;\boldsymbol{\tau}_{P-1})$ is a complex sinusoid with normalized frequency [5]

$$f = 2^{P-1}\Delta^P P! \alpha_P \prod_{k=1}^{P-1} \tau_k. \quad (4)$$

The coefficient α_P can therefore be estimated by searching for the position of maximum in the ml-HAF. Once the estimation of α_P, denoted as $\widehat{\alpha}_P$, is obtained, we can demodulate $x(n)$ by $\exp(-j2\pi\widehat{\alpha}_P(n\Delta)^P)$ to reduce the PPS order by one. The procedure is repeated until all remaining coefficients are estimated [2].

In order to avoid aliasing in estimating α_P, the following relation must hold:

$$|\alpha_P| \le \alpha_{P\,\max}^{\tau} = \frac{1}{2^P P! \Delta^P \prod_{k=1}^{P-1} \tau_k}, \quad (5)$$

since f is limited to $[-\frac{1}{2}, \frac{1}{2})$.

In [11], the authors proposed a method to recover the true PPS parameters if aliasing occurred. To that end, two HAFs are calculated, using coprime lags τ_1 and τ_2, from which two (possibly aliased) peak locations f_1 and f_2, where $f_1, f_2 \in [-\frac{1}{2}, \frac{1}{2})$, are obtained. Therefore, two integers k_1 and k_2 exist such that[1]

$$\begin{aligned} k_1 + f_1 &= 2^{P-1}\Delta^P P! \alpha_P \tau_1^{P-1} \\ k_2 + f_2 &= 2^{P-1}\Delta^P P! \alpha_P \tau_2^{P-1}. \end{aligned} \quad (6)$$

Combining the equations in (6) yields the linear Diophantine equation

$$k_2\tau_1^{P-1} - k_1\tau_2^{P-1} = f_1\tau_2^{P-1} - f_2\tau_1^{P-1} \triangleq M. \quad (7)$$

In the following step, two integers, n_1 and n_2, such that $n_2\tau_1^{P-1} - n_1\tau_2^{P-1} = 1$ are found. The solutions k_1 and k_2 of (7) are completely characterized by [11]

$$\begin{aligned} k_1 &= n_1 M + q\tau_1^{P-1} \\ k_2 &= n_2 M - q\tau_2^{P-1}, \end{aligned} \quad (8)$$

where $q \in \mathbf{Z}$. By changing q we find a value of k_1, denoted as k_1^*, such that $k_1 + f_1$ falls within $[-\frac{\tau_1^{P-1}}{2}, \frac{\tau_1^{P-1}}{2})$. Finally, the parameter α_P is estimated as

$$\alpha_P = \frac{k_1^* + f_1}{2^{P-1}\Delta^P P! \tau_1^{P-1}}. \quad (9)$$

In the rest of the paper, we will refer to the method proposed in [11] as the ZW method.

The problem with the ZW method is the estimation of number M in (7). Due to the errors in estimation of f_1 and f_2, M will not be an integer and it has to be rounded to the nearest integer [11]. If we assume that the errors in

[1] Note that, comparing to (4) and (5) in [11], in (6) we have an additional term 2^{P-1} which is due to the symmetric definition of the HIM that we use. In addition, in [11], $\Delta = 1$ and, in the HIM calculation, all lags coincide.

estimation of f_1 and f_2 are uncorrelated and characterized by zero mean and same variance σ_f^2, the variance of estimation of M equals

$$\sigma_M^2 = \left(\tau_1^{2(P-1)} + \tau_2^{2(P-1)}\right)\sigma_f^2. \quad (10)$$

Clearly, σ_M^2 can take significant values with higher values of P, which, in turn, can lead to false rounded value of M used in (8). This is illustrated in Figs. 1b) and 2b) in the Simulations section.

In addition, in the ZW method, we cannot conclude whether aliasing occurred or not based only on the peak locations f_1 and f_2. The resolving procedure is performed anyway and the obtained value k_1^* indicates aliasing, i.e., aliasing occurred if $k_1^* \neq 0$.

In the following section, we first propose how to detect parameter aliasing, and, if aliasing occurred, how to recover the true parameter value without solving Diophantine equations.

III. Aliasing detection and resolving

Let us assume that restriction (5) does not apply to α_P. Then α_P can be written as

$$\alpha_P = 2Q\alpha_{P\,\max}^{\boldsymbol{\tau}} + \Delta\alpha_P, \quad (11)$$

where $Q = 0, \pm1, \pm2, \pm3, ...$ and $\Delta\alpha_P$ is residual that satisfies $|\Delta\alpha_P| < \alpha_{P\,\max}^{\boldsymbol{\tau}}$. The case of no aliasing corresponds to $Q = 0$, whereas all other Qs imply aliasing. After calculating the Pth order ml-HAF (3), a spectral peak will appear at frequency

$$f = 2^{P-1}\Delta^P P!\Delta\alpha_P \prod_{k=1}^{P-1} \tau_k. \quad (12)$$

The estimation of $\Delta\alpha_P$ does not suffice for determining α_P since Q remains unknown.

Consider first the case of no aliasing. If we use the other set of time lags, $\boldsymbol{\tau'}_{P-1} = [\tau'_1, \tau'_2, \cdots, \tau'_{P-1}]$, in calculating the Pth order ml-HIM, and calculate the ml-HAF $X_P(S(\boldsymbol{\tau'}_{P-1}, \boldsymbol{\tau}_{P-1})f; \boldsymbol{\tau'}_{P-1})$, where the scaling coefficient $S(\boldsymbol{\tau'}_{P-1}, \boldsymbol{\tau}_{P-1})$ satisfies

$$S(\boldsymbol{\tau'}_{P-1}, \boldsymbol{\tau}_{P-1}) = \prod_{k=1}^{P-1} \frac{\tau'_k}{\tau_k}, \quad (13)$$

a spectral peak will appear at frequency [5]

$$f' = 2^{P-1}\Delta^P P!\Delta\alpha'_P \prod_{k=1}^{P-1} \tau_k, \quad (14)$$

which coincides with (12) since $\Delta\alpha_P = \Delta\alpha'_P$ and all other terms are the same. This is the well-known product HAF (PHAF) principle of aligning the autoterms in frequency [5]. Therefore, when no aliasing occurs, $X_P(f; \boldsymbol{\tau}_{P-1})$ and $X_P(S(\boldsymbol{\tau'}_{P-1}, \boldsymbol{\tau}_{P-1})f; \boldsymbol{\tau'}_{P-1})$ will have a spectral peak at the same frequency.

Consider now the case of aliasing. According to (5) and (11), using a different set of time lags $\boldsymbol{\tau'}_{P-1}$ yields $\alpha_{P\,\max}^{\boldsymbol{\tau'}}$ and $\Delta\alpha'_P$ that differ from $\alpha_{P\,\max}^{\boldsymbol{\tau}}$ and $\Delta\alpha_P$ obtained when $\boldsymbol{\tau}_{P-1}$ is used. In addition, the corresponding value of Q' is possibly different from Q. However, $\alpha_{P\,\max}^{\boldsymbol{\tau'}}$, $\Delta\alpha'_P$ and Q' satisfy

$$\alpha_P = 2Q'\alpha_{P\,\max}^{\boldsymbol{\tau'}} + \Delta\alpha'_P. \quad (15)$$

The difference between $\boldsymbol{\tau}_{P-1}$ and $\boldsymbol{\tau'}_{P-1}$ gives rise to different positions of spectral peaks in $X_P(f; \boldsymbol{\tau}_{P-1})$ and $X_P(S(\boldsymbol{\tau'}_{P-1}, \boldsymbol{\tau}_{P-1})f; \boldsymbol{\tau'}_{P-1})$, since the peak frequency depends on residual $\Delta\alpha_P$.

Therefore, the aliasing detection is performed by comparing the peak frequencies of HAFs $X_P(f; \boldsymbol{\tau}_{P-1})$ and $X_P(S(\boldsymbol{\tau'}_{P-1}, \boldsymbol{\tau}_{P-1})f; \boldsymbol{\tau'}_{P-1})$, namely f and f', respectively. If the peak frequencies coincide, aliasing has not occurred. Otherwise, it has occurred.

In addition to detection, the values of f and f' can serve in recovering the true value of α_P if aliasing occurred. Assume, for the moment, that $\prod_{k=1}^{P-1} \tau_k$ and $\prod_{k=1}^{P-1} \tau'_k$ do not differ much, so that $Q = Q'$ holds. In that case, combining (11) and (15) gives

$$Q = \frac{\Delta\alpha_P - \Delta\alpha'_P}{2(\alpha_{P\,\max}^{\boldsymbol{\tau'}} - \alpha_{P\,\max}^{\boldsymbol{\tau}})},$$

which combined with (5), (12) and (14) gives the final value of Q as

$$Q = \frac{f - f'}{\prod_{k=1}^{P-1} \frac{\tau_k}{\tau'_k} - 1}. \quad (16)$$

The estimated Q has to be rounded to the closest integer and the true value of α_P can now be estimated either from (11) or (15). Using (11) gives the final expression for α_P as

$$\alpha_P = \frac{f - f' \prod_{k=1}^{P-1} \frac{\tau'_k}{\tau_k}}{2^{P-1}\Delta^P P! \left(\prod_{k=1}^{P-1} \tau_k - \prod_{k=1}^{P-1} \tau'_k\right)}. \quad (17)$$

If the values of $\prod_{k=1}^{P-1} \tau_k$ and $\prod_{k=1}^{P-1} \tau'_k$ differ so much that $Q = Q'$ does not hold, relation (16) will not be correct and the values of α_P obtained from (11) and (15) will not coincide. In that case, we have to decrease the difference between these two products. Specifically, we can choose lags τ_k close to the optimal values[2] [5], and lags τ'_k to satisfy $\tau'_k = \tau_k - L$, where L is a predefined integer. If $Q = Q'$ does not hold, we can reduce L by one. If, however, $L = 1$, we can decrease τ_k, set L to the predefined value and repeat the procedure. Decreasing τ_k provides wider dynamic range of α_P (5).

The estimation procedure can be summarized as follows.

Step 1. Calculate the HAFs $X_P(f; \boldsymbol{\tau}_{P-1})$ and $X_P(S(\boldsymbol{\tau'}_{P-1}, \boldsymbol{\tau}_{P-1})f; \boldsymbol{\tau'}_{P-1})$ and find the corresponding peak frequencies f and f', respectively. If $f = f'$ aliasing has not occurred. Estimate α_P according to (4) and exit. Otherwise, go to Step 2.

Step 2. Determine Q according to (16) and estimate α_P using (11) and (15), where $\Delta\alpha_P$ and $\Delta\alpha'_P$ are obtained from (12) and (14), respectively. If the obtained α_P values coincide ($Q = Q'$ holds), the estimation is correct and exit the procedure. Otherwise, go to Step 3.

Step 3. Decrease the difference between $\prod_{k=1}^{P-1} \tau_k$ and $\prod_{k=1}^{P-1} \tau'_k$ and go to Step 1.

In case of multicomponent PPSs, the extension of the proposed algorithm is straightforward. We can use the approach proposed in [3, Section II], where two steps, namely 1) the estimation of parameters of the strongest component and 2) filtering that component out, are iterated N_c times, where N_c is the number of components. In the current iteration, the parameters of the current strongest component are estimated, starting from the highest one, using the procedure summarized above. Therefore, the check for aliasing and alias resolving if it occurred are performed for each parameter of each component of multicomponent PPS.

[2]The optimal lags for the Pth order ml-HAF are all equal to each other and to

$$\tau_{opt} = \frac{N}{2P}. \quad (18)$$

The optimality criterion is the resolution capability. Relation (18) is equivalent to the one suggested in [1] for the special cases $P = 2, 3$ and $\tau_1 = \tau_2 = \cdots \tau_{P-1}$, where the optimality criterion is the variance of the estimates.

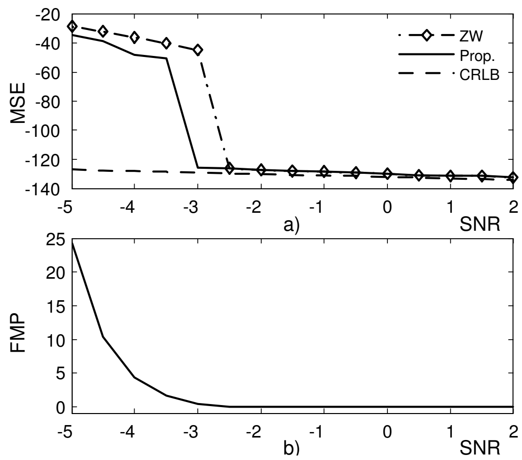

Fig. 1. The α_2 estimation. a) MSE versus SNR (Prop. stands for proposed method). b) False M percentage (FMP) versus SNR for the ZW method.

IV. Simulations

In this section, we will estimate the parameters of a PPS $x(n)$ from

$$y(n) = x(n) + \nu(n), \quad n = 0, \cdots, N-1, \quad (19)$$

where $\nu(n)$ is zero-mean complex Gaussian noise with i.i.d. real and imaginary parts, and variance σ_ν^2. The signal-to-noise ratio (SNR) is defined as $\text{SNR} = 10\log_{10}(A^2/\sigma_\nu^2)$. In addition, $N = 512$ and $\Delta = 1$. We compared our method to the ZW method in terms of the mean squared error (MSE), calculated over 500 trials.

Example 1. Let us first consider the parameter estimation of a single chirp $x(n) = A\exp(j2\pi\alpha_2(n\Delta)^2)$, where $\alpha_2 = 9.73 \times 10^{-3}$. Note that this chirp is undersampled; its bandwidth exceeds the sampling rate around ten

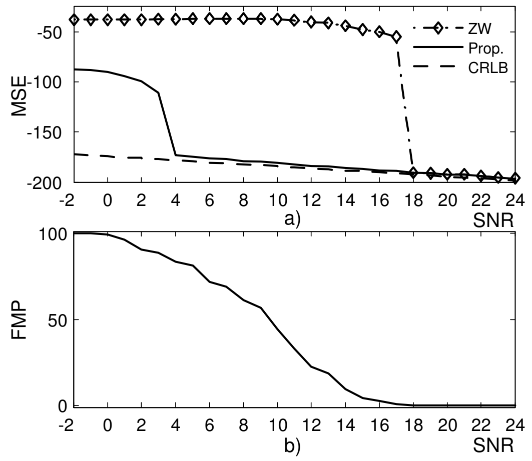

Fig. 2. The α_3 estimation. a) MSE versus SNR (Prop. stands for proposed method). b) False M percentage (FMP) versus SNR for the ZW method.

times. In the proposed method, we used lags $\tau_1 = 128$ and $\tau'_1 = 126$. On the other hand, in the ZW method, we used coprime lags $\tau_1 = 128$ and $\tau_2 = 121$. With these lags, we have $M = Q = 5$ (see (7) and (11)). The obtained MSE versus SNR curves are depicted in Fig. 1a), where the SNR is varied from -5 dB to 2 dB in increments of 0.5 dB. Our method has a bit lower SNR threshold, approximately 0.5 dB. The Cramér-Rao lower bound (CRLB) is also given in Fig. 1a). In addition to the MSE curves, in Fig. 1b), we depicted the false M percentage (FMP), where M is obtained from (7) and used in (8). Clearly, the SNR threshold coincides with the appearance of false Ms, which is around SNR $= -3$ dB.

Example 2. Now we consider a cubic phase signal $x(n) = A\exp(j2\pi(\alpha_1(n\Delta) + \alpha_2(n\Delta)^2 + \alpha_3(n\Delta)^3))$, where $\alpha_1 = 0.491$, $\alpha_2 = -3.79 \times 10^{-3}$ and $\alpha_3 = 4.91 \times 10^{-6}$, and the estimation of the coefficient α_3. In the proposed method, we used lag sets $\boldsymbol{\tau}_2 = [86, 86]$ and $\boldsymbol{\tau'}_2 = [86, 82]$. In the ZW method, we used coprime lags $\tau_1 = 85$ and $\tau_2 = 81$. With these lags, we have $M = Q = 1$. Note that $x(n)$ is not undersampled, although its coefficients α_2 and α_3 are aliased. The obtained MSE versus SNR curves are depicted in Fig. 2a), where the SNR is varied from -2 dB to 24 dB in increments of 1 dB. Now, the difference in performance is tremendous. Our method

TABLE I

MSE OF PARAMETER ESTIMATION OF A TWO-COMPONENT PPS

	Proposed method	ZW method
α_{12}	-119.12 dB	-22.75 dB
α_{13}	-176.89 dB	-39.96 dB
α_{22}	-121.83 dB	-23.04 dB
α_{23}	-179.36 dB	-39.38 dB

has the SNR threshold that is around 14 dB lower than that of the ZW method. Again, the FMP curve, given in Fig. 2b), proves that ZW method's poor performance is due to the bad estimation of M used in (8). False Ms begin to appear around SNR $= 17$ dB, which corresponds to the SNR threshold for the ZW method.

Note that in both the second- and third-order parameter estimation, our method retains the SNR threshold determined for the HAF when PPS with non-aliased parameters is considered [14]. Specifically, for the second-order, the threshold is -3 dB, whereas for the third-order it is 3.77 dB (see Table II in [14]).

Example 3. Finally, we consider a two-component signal

$$x(n) = A_1 e^{(j2\pi(\alpha_{12}(n\Delta)^2 + \alpha_{13}(n\Delta)^3))} + A_2 e^{(j2\pi(\alpha_{22}(n\Delta)^2 + \alpha_{23}(n\Delta)^3))},$$

where $(\alpha_{12}, \alpha_{13}) = (2.59 \times 10^{-4}, 4.71 \times 10^{-5})$ and $(\alpha_{22}, \alpha_{23}) = (9.73 \times 10^{-3}, 2.19 \times 10^{-6})$. In addition, $10\log_{10}(A_1^2/\sigma_\nu^2) = 13$ dB and $10\log_{10}(A_2^2/\sigma_\nu^2) = 10$ dB. In the proposed method, we used $\boldsymbol{\tau}_2 = [86, 86]$, and $\boldsymbol{\tau'}_2 = [86, 82]$ for the third-order HIM, and $\tau_1 = 128$ and $\tau'_1 = 126$ for the second-order HIM. In the ZW method, we used $\tau_1 = 85$ and $\tau_2 = 81$ for the third-order HIM and $\tau_1 = 128$ and $\tau_2 = 121$ for the second order HIM. With these lag values, parameters α_{12} and α_{23} are not aliased, whereas α_{13} and α_{22} are aliased with $M = Q = 5$ and $M = Q = 8$, respectively. The obtained MSE values in the estimation of both components' parameters are given in Table I.

As opposed to the ZW method, our method successfully resolved aliasing of both components' parameters. The ZW method fails to resolve aliasing when components are PPSs of third or higher order. The influence of components on each other increases the variance of the frequency estimation σ_f^2, thus making the estimation of M less accurate.

V. Conclusions

In this paper, we proposed the method for aliasing detection and resolving in the estimation of PPS parameters. Both detection and resolving of aliasing are performed using maxima positions of two HAFs of the PPS embedded in noise. We derived the expression for calculating the true value of the aliased PPS parameter. The proposed method provides much more robust results than the one proposed in [11] when higher order PPSs are considered. Furthermore, it retains the SNR threshold derived for the case when no aliasing occurs and it does not require a nonuniform sampling as the one proposed in [13]. The method can be readily extended to deal with multicomponent PPSs.

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