Analysis of Noise in Time-Frequency Distributions

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Noise analysis for commonly used time-frequency distributions is presented. The Wigner distribution, as a basic time-frequency representation, is studied first. The bias and variance in the case of complex white noise are derived. The analysis of noise is extended to other quadratic distributions, and to different types of additive and multiplicative noise, including: stationary white noise, nonstationary white noise, and colored stationary noise. Exact expressions for the mean value and the variance of quadratic distributions for each point in the time-frequency plane are given.

A. Wigner distribution

The pseudo Wigner distribution (WD) of a discrete-time noisy signal $x(n) = s(n) + \epsilon(n)$ is defined by[1]:

$$W_x(n,f) = \sum\nolimits_m w(m)w(-m)x(n+m)x^*(n-m)e^{-j4\pi fm}. \tag{1}$$

where $w(m)$ is a real-valued lag window, such that $w(0) = 1$.

Consider first the case when $s(n)$ is deterministic and the noise $\epsilon(n)$ is a white, Gaussian, complex, stationary, zero-mean process, with independent real and imaginary parts having equal variances. Its autocorrelation function is $R_{\epsilon\epsilon}(m) = \sigma_\epsilon^2\delta(m)$. The WD mean for the noisy signal $x(n)$ is

$$E\{W_x(n,f)\} = \sum\nolimits_m w(m)w(-m)s(n+m)s^*(n-m)e^{-j4\pi fm} + \sum\nolimits_m w(m)w(-m)R_{\epsilon\epsilon}(2m)e^{-j4\pi fm}$$

$$= 2\int_{-1/4}^{1/4} W_s(n,f-\alpha)F_w(2\alpha)d\alpha + \sigma_\epsilon^2, \tag{2}$$

where $F_w(f) = \mathcal{F}_{m\to f}[w(m)w(-m)]$ is the Fourier transform (FT) of the product $w(m)w(-m)$, and $W_s(n,f)$ is the original WD of $s(n)$, without a lag window.

The lag window $w(m)$ causes the **WD bias**. The second term on the right-hand side in (2) is constant, so one can assume that it does not distort the WD. Expanding $W_s(n,f-\alpha)$ into a Taylor series, around f, we get

$$2\int_{-1/4}^{1/4} W_s(n,f-\alpha)F_w(2\alpha)d\alpha \cong W_s(n,f) + \frac{1}{8}\frac{\partial^2 W_s(n,f)}{\partial f^2}m_2 + ... \tag{3}$$

Thus, the bias can be approximated by

$$\text{bias}\,(n,f) \cong \frac{1}{8}\frac{\partial^2 W_s(n,f)}{\partial f^2}m_2 = \frac{1}{8}b(n,f)m_2,$$

where $m_2 = \int_{-1/2}^{1/2} f^2 F_w(f)df$. For the regions where the WD variations in the frequency direction are small, the bias is small, and vice versa.

The WD estimator **variance**, at a given point (n,f), is defined by:

$$\sigma_{WD}^2(n,f) = E\{W_x(n,f)W_x^*(n,f)\} - E\{W_x(n,f)\}E\{W_x^*(n,f)\}. \tag{4}$$

For signals $x(n) = s(n) + \epsilon(n)$ it results in

$$\sigma_{WD}^2(n,f) = \sum_{m_1}\sum_{m_2} w(m_1)w(-m_1)w(m_2)w(-m_2)e^{-j4\pi f(m_1-m_2)}$$
$$\times[s(n+m_1)s^*(n+m_2)R_{\epsilon\epsilon}(n-m_2,n-m_1)$$
$$+s^*(n-m_1)s(n-m_2)R_{\epsilon\epsilon}(n+m_1,n+m_2)$$

Time-Frequency Signal Analysis and Processing, ed. B. Boashash, Elsevier, 2003.

[1]Notation $\sum_m$, without limits, will be used for $\sum_{m=-\infty}^{\infty}$. The constant factor of 2 is omitted in the WD definition, and in other TFD definitions.

$$+s(n+m_1)s(n-m_2)R^*_{\epsilon\epsilon^*}(n-m_1,n+m_2)$$

$$+s^*(n-m_1)s^*(n+m_2)R_{\epsilon\epsilon^*}(n+m_1,n-m_2)$$

$$+R_{\epsilon\epsilon}(n+m_1,n+m_2)R_{\epsilon\epsilon}(n-m_2,n-m_1)$$

$$+R_{\epsilon\epsilon^*}(n+m_1,n-m_2)R^*_{\epsilon\epsilon^*}(n-m_1,n+m_2)]. \tag{5}$$

The fourth-order moment of noise is reduced to the correlation functions by using the relation $E\{z_1z_2z_3z_4\} = E\{z_1z_2\}E\{z_3z_4\} + E\{z_1z_3\}E\{z_2z_4\} + E\{z_1z_4\}E\{z_2z_3\}$, which holds for Gaussian zero-mean random variables z_i, $i = 1,2,3,4$. For the considered complex noise $R_{\epsilon\epsilon}(n,m) = \sigma_\epsilon^2\delta(n-m)$ and $R_{\epsilon\epsilon^*}(n,m) = 0$. The variance of the WD estimator reduces to

$$\sigma^2_{WD}(n,f) = \sigma^2_\epsilon \sum\nolimits_m w^2(m)w^2(-m)$$

$$\times \left[2\left|s(n+m)\right|^2 + \sigma^2_\epsilon\right].$$

It is frequency independent. For constant modulus signals, $s(n) = a\exp[j\phi(n)]$, the variance is constant $\sigma^2_{WD}(n,f) = \sigma^2_\epsilon E_w(2a^2+\sigma^2_\epsilon)$, where $E_w = \sum_m [w(m)w(-m)]^2$ is the energy of $w(m)w(-m)$ window. A finite energy lag window is sufficient to make the variance of $W_x(n,f)$ finite.

The optimal lag window width can be obtained by minimizing the error $e^2 = \mathrm{bias}^2(n,f) + \sigma^2_{WD}(n,f)$. For example, for constant modulus signals, and the Hanning window $w(m)w(-m)$ of the width N, when $E_w = 3N/8$ and $m_2 = 1/(2N^2)$, we get:

$$e^2 \cong \frac{1}{256N^4}b^2(n,f) + \frac{3N}{8}\sigma^2_\epsilon(2a^2+\sigma^2_\epsilon).$$

From $\partial e^2/\partial N = 0$ the approximation of optimal window width follows:

$$N_{opt}(n,f) \cong \sqrt[5]{\frac{b^2(n,f)}{24\sigma^2_\epsilon(2a^2+\sigma^2_\epsilon)}}.$$

An approach to the calculation of the estimate of $N_{opt}(n,f)$, without using the value of $b^2(n,f)$, is presented in [10], [Article 10.2]. Other statistical properties of the Wigner distribution are studied in [4].

B. *Noise in Quadratic Time-Frequency Distributions*

A discrete-time form of the Cohen class of distributions of noise $\epsilon(n)$ is defined by:

$$\rho_\epsilon(n,f;G) =$$

$$\sum\nolimits_l \sum\nolimits_m G(m,l)\epsilon(n+m+l)\epsilon^*(n+m-l)e^{-j4\pi fl}, \tag{6}$$

where $G(m,l)$ is the kernel in the time-lag domain.

Its **mean value**, for a general nonstationary noise, is

$$E\{\rho_\epsilon(n,f;G)\} =$$

$$\sum\nolimits_l \sum\nolimits_m G(m,l)R_{\epsilon\epsilon}(n+m+l,n+m-l)e^{-j4\pi fl},$$

where $R_{\epsilon\epsilon}(m,n)$ is the noise autocorrelation function. For special cases of noise the values of $E\{\rho_\epsilon(n,f;G)\}$ follow.

1) Stationary white noise, $R_{\epsilon\epsilon}(m,n) = \sigma^2_\epsilon\delta(m-n)$,

$$E\{\rho_\epsilon(n,f;G)\} = \sigma^2_\epsilon g(0,0).$$

2) Nonstationary white noise, $R_{\epsilon\epsilon}(m,n) = I(n)\delta(m-n)$, $I(n) \geq 0$,

$$E\{\rho_\epsilon(n,f;G)\} = \sum\nolimits_m G(m,0)I(n+m).$$

3) Stationary colored noise, $R_{\epsilon\epsilon}(m,n) = R_{\epsilon\epsilon}(m-n)$,

$$E\{\rho_\epsilon(n,f;G)\} = \int_{-1/2}^{1/2} \mathcal{G}(0,2(f-\alpha))S_{\epsilon\epsilon}(\alpha)d\alpha,$$

where $S_{\epsilon\epsilon}(f) = \mathcal{F}_{m\to f}[R_{\epsilon\epsilon}(m)]$ is the noise power spectrum density, and the kernel forms in time-lag, Doppler-lag, and Doppler-frequency domains are denoted by:

$$\sum\nolimits_m G(m,l)e^{-j2\pi\nu m} = g(\nu,l)$$

$$= \int_{-1/2}^{1/2} \mathcal{G}(\nu,f)e^{j2\pi fl}df. \tag{7}$$

The **variance** of $\rho_\epsilon(n,f;G)$, is defined by

$$\sigma^2_{\epsilon\epsilon}(n,f) = E\{\rho_\epsilon(n,f;G)\rho^*_\epsilon(n,f;G)\}$$

$$-E\{\rho_\epsilon(n,f;G)\}E\{\rho^*_\epsilon(n,f;G)\}.$$

For Gaussian noise, as in (4)-(5), we get:

$$\sigma^2_{\epsilon\epsilon}(n,f) =$$

$$\sum\nolimits_{l_1}\sum\nolimits_{l_2}\sum\nolimits_{m_1}\sum\nolimits_{m_2} G(m_1,l_1)G^*(m_2,l_2)$$
$$\times[R_{\epsilon\epsilon}(n+m_1+l_1,n+m_2+l_2)$$
$$\times R^*_{\epsilon\epsilon}(n+m_1-l_1,n+m_2-l_2)$$
$$+R_{\epsilon\epsilon^*}(n+m_1+l_1,n+m_2-l_2)$$
$$\times R_{\epsilon^*\epsilon}(n+m_1-l_1,n+m_2+l_2)]e^{-j4\pi f(l_1-l_2)}. \tag{8}$$

Form of $\sigma^2_{\epsilon\epsilon}(n,f)$ for the specific noises will be presented next.

Complex stationary and nonstationary white noise

For nonstationary complex white noise, with independent real and imaginary part of equal variance, $R_{\epsilon\epsilon}(m,n) = I(n)\delta(m-n)$, $R_{\epsilon\epsilon^*}(n,m)=0$, we get

$$\sigma^2_{\epsilon\epsilon}(n,f) =$$

$$\sum\nolimits_{l}\sum\nolimits_{m} |G(m,l)|^2 I(n+m+l)I^*(n+m-l)$$
$$=\rho_I(n,0;|G|^2). \tag{9}$$

For stationary white noise, $I(n)=\sigma^2_\epsilon$, the variance is proportional to the kernel energy,

$$\sigma^2_{\epsilon\epsilon}(n,f) = \sigma^4_\epsilon \sum\nolimits_{l}\sum\nolimits_{m} |G(m,l)|^2. \tag{10}$$

Colored stationary noise

For complex colored stationary noise, the variance (8) can be written as

$$\sigma^2_{\epsilon\epsilon}(n,f) =$$

$$\sum\nolimits_{l_1}\sum\nolimits_{m_1} G(m_1,l_1)\sum\nolimits_{m_2}\sum\nolimits_{l_2} G^*(m_2,l_2)$$
$$\times[R_{\epsilon\epsilon}(m_1-m_2+l_1-l_2)R^*_{\epsilon\epsilon}(m_1-m_2-(l_1-l_2))]$$
$$\times e^{-j4\pi f(l_1-l_2)},$$

or

$$\sigma^2_{\epsilon\epsilon}(n,f) = \sum\nolimits_{l}\sum\nolimits_{m} G(m,l)$$
$$\times\{G(m,l) *_l *_m [R^*_{\epsilon\epsilon}(m+l)R_{\epsilon\epsilon}(m-l)e^{j4\pi fl}]\}^*,$$

where " $*_l *_m$" denotes a twodimensional convolution in l,m. Consider the product of $G(m,l)$ and $Y^*(m,l) = \{G(m,l) *_l *_m [R^*_{\epsilon\epsilon}(m+l)R_{\epsilon\epsilon}(m-l)e^{j4\pi fl}]\}^*$ in the last expression. Twodimensional FTs of these terms are $\mathcal{G}(\nu,\xi)$ and $y(\nu,\xi) = \mathcal{G}(\nu,\xi)S_{\epsilon\epsilon}(f-(\xi-\nu)/2)S^*_{\epsilon\epsilon}(f-(\xi+\nu)/2)/2$. According to the Parseval's theorem we get:

$$\sigma^2_{\epsilon\epsilon}(n,f) = \frac{1}{2}\int_{-1/2}^{1/2}\int_{-1/2}^{1/2} |\mathcal{G}(\nu,\xi)|^2$$
$$\times S^*_{\epsilon\epsilon}(f-\frac{\xi}{2}+\frac{\nu}{2})S_{\epsilon\epsilon}(f-\frac{\xi}{2}-\frac{\nu}{2})d\nu d\xi$$
$$=\rho_{S_{\epsilon\epsilon}}(0,f;|\mathcal{G}|^2), \tag{11}$$

for $|f-(\xi-\nu)/2| < 1/2$ and $|f-(\xi+\nu)/2| < 1/2$. The transforms in (11) are periodic in ν and ξ with period 1. It means that we should take into account all ν and ξ when $|f-[(\xi+k_1)-(\nu+k_2]/2| < 1/2$ and $|f-[(\xi+k_1)+(\nu+k_2)]/2| < 1/2$, where k_1 and k_2 are integers.

Note that the FT of a colored stationary noise is a white nonstationary noise, with autocorrelation in the frequency domain

$$R_{\Xi\Xi}(f_1,f_2) =$$

$$\sum\nolimits_{m}\sum\nolimits_{n} E\{\epsilon(m)\epsilon^*(n)\}e^{(-j2\pi f_1 m+j2\pi f_2 n)}$$
$$=S_{\epsilon\epsilon}(f_2)\delta_p(f_1-f_2),$$

where $\delta_p(f)$ is a periodic delta function with period 1. Thus, (11) is just a form dual to (9).

Analytic noise

In the numerical implementation of quadratic distributions, an analytic part of the signal is commonly used, rather than the signal itself. The analytic part of noise can be written as $\epsilon_a(n) = \epsilon(n) + j\epsilon_h(n)$, where $\epsilon_h(n) = \mathcal{H}[\epsilon(n)]$ is the Hilbert transform of $\epsilon(n)$. Spectral power density of $\epsilon_a(n)$, within the basic period $|f| < 1/2$, for a white noise $\epsilon(n)$, is $S_{\epsilon_a\epsilon_a}(f) = 2\sigma^2_\epsilon U(f)$, where $U(f)$ is the unit step function. The variance follows from (11) in the form

$$\sigma^2_{\epsilon\epsilon}(n,f) = 2\sigma^4_\epsilon$$

$$\times\int_{-1/2}^{1/2}\int_{-d(f,\xi)}^{d(f,\xi)} |\mathcal{G}(\nu,\xi)|^2 d\xi d\nu \quad for \quad |2f| \le \frac{1}{2}, \tag{12}$$

where the integration limits are defined by $d(f,\xi) = |\arcsin(\sin(\pi(2f-\xi)))|/\pi$ (for details see [8]).

The kernel $\mathcal{G}(\nu,\xi)$ is mainly concentrated at and around the (ν,ξ) origin and $\xi = 0$ axis. Having this in mind, as well as the fact that $|\mathcal{G}(\nu,\xi)|^2$ is always positive, we may easily conclude that the minimal value of $\sigma^2_{\epsilon\epsilon}(n,f)$ is for $f = 0$. The maximal value will be obtained for $|f| = 1/4$. It is very close to [8]:

$$\max\{\sigma^2_{\epsilon\epsilon}(n,f)\} \cong 2\sigma^4_\epsilon \int_{-1/2}^{1/2}\int_{-1/2}^{1/2} |\mathcal{G}(\nu,\xi)|^2\, d\xi d\nu$$

$$= 2\sigma^4_\epsilon \sum\nolimits_l \sum\nolimits_m |G(m,l)|^2 .$$

Real noise

Now consider a real stationary white Gaussian noise $\epsilon(n)$ with variance σ^2_ϵ. In this case, variance (8) contains all terms. It can be written as:

$$\sigma^2_{\epsilon\epsilon}(n,f) = \sigma^4_\epsilon \sum\nolimits_l \sum\nolimits_m [|G(m,l)|^2$$

$$+ G(m,l)G^*(m,-l)e^{-j8\pi fl}]. \qquad (13)$$

For distributions whose kernel is symmetric with respect to l, $G(m,l) = G(m,-l)$ holds. The FT is therefore applied to the positive and even function $|G(m,l)|^2$. The transform's maximal value is reached at $f = 0$, and $|f| = 1/4$. Accordingly:

$$\max\left\{\sigma^2_{\epsilon\epsilon}(n,f)\right\} = 2\sigma^4_\epsilon \sum\nolimits_l \sum\nolimits_m |G(m,l)|^2 . \qquad (14)$$

The crucial parameter in all previous cases is the kernel energy $\sum_l \sum_m |G(m,l)|^2$. Its minimization is thoroughly studied in [1]. It has been concluded that, out of all the quadratic distributions satisfying the marginal and time-support conditions, the Born-Jordan distribution is optimal with respect to this parameter.

C. Noisy Signals

Analysis of deterministic signals $s(n)$ corrupted by noise, $x(n) = s(n) + \epsilon(n)$, is highly signal dependent. It can be easily shown [1], [8], that the distribution variance $\sigma^2_\rho(n,f)$ consists of two components:

$$\sigma^2_\rho(n,f) = \sigma^2_{\epsilon\epsilon}(n,f) + \sigma^2_{s\epsilon}(n,f). \qquad (15)$$

The first variance component, and the distribution mean value, have already been studied in detail.[2] For the analysis of the second, signal dependent, component $\sigma^2_{s\epsilon}(n,f)$ we will use the inner product form of the Cohen class of distributions:

$$\rho_x(n,f;\tilde{G}) = \sum\nolimits_l \sum\nolimits_m \tilde{G}(m,l)$$

$$\times \left[x(n+m)e^{-j2\pi fm}\right]\left[x(n+l)e^{-j2\pi fl}\right]^*, \quad (16)$$

where $\tilde{G}(m,l) = G((m+l)/2,(m-l)/2)$. Calculation of $\tilde{G}(m,l)$ is described in the next section. For a real and symmetric $G(m,l)$, and complex noise, we get

$$\sigma^2_{s\epsilon}(n,f) =$$

$$2\sum_{l_1}\sum_{m_1}\sum_{l_1}\sum_{m2} \tilde{G}(m_1,l_1)\tilde{G}^*(m_2,l_2)s(n+m_1)$$

$$\times s^*(n+m_2)R_{\epsilon\epsilon}(n+l_2,n+l_1)e^{-j2\pi f(m_1-l_1-m_2+l_2)},$$

what can be written as

$$\sigma^2_{s\epsilon}(n,f) = 2\sum\nolimits_{m_1}\sum\nolimits_{m_2} \tilde{\Phi}(m_1,m_2)$$

$$\times[s(n+m_1)e^{-j2\pi fm_1}][s(n+m_2)e^{-j2\pi fm_2}]^*, \qquad (17)$$

where the new kernel $\tilde{\Phi}(m_1,m_2)$ reads

$$\tilde{\Phi}(m_1,m_2) = \sum\nolimits_{l_1}\sum\nolimits_{l_2} \tilde{G}(m_1,l_1)\tilde{G}^*(m_2,l_2)$$

$$\times e^{-j2\pi f(l_2-l_1)}R_{\epsilon\epsilon}(n+l_2,n+l_1). \qquad (18)$$

The signal dependent part of the variance $\sigma^2_\rho(n,f)$ is a quadratic distribution of the signal, with the new kernel $\tilde{\Phi}(m_1,m_2)$, i.e., $\sigma^2_{s\epsilon}(n,f) = 2\rho_s(n,f;\tilde{\Phi})$.

Special case 1: **White stationary complex noise**, when $R_{\epsilon\epsilon}(n+l_1,n+l_2) = \sigma^2_\epsilon\delta(l_1-l_2)$, produces

$$\tilde{\Phi}(m_1,m_2) = \sigma^2_\epsilon \sum\nolimits_l \tilde{G}(m_1,l)\tilde{G}^*(m_2,l). \quad (19)$$

For time-frequency kernels we assumed realness and symmetry throughout the article, i.e., $\tilde{G}^*(m_2,l) = \tilde{G}(l,m_2)$. Thus, for finite limits (19) is a matrix multiplication form,

[2] An analysis of the bias, i.e., kernel influence on the form of $\rho_s(n,f;G)$ may be found in [10].

$\tilde{\mathbf{\Phi}} = \sigma_\epsilon^2 \tilde{\mathbf{G}} \cdot \tilde{\mathbf{G}}^* = \sigma_\epsilon^2 \tilde{\mathbf{G}}^2$. Boldface letters, without arguments, will be used to denote a matrix. For example $\tilde{\mathbf{G}}$ is a matrix with elements $\tilde{G}(m,l)$. Thus,

$$\sigma_{s\epsilon}^2(n,f) = 2\rho_s(n,f;\sigma_\epsilon^2 \tilde{\mathbf{G}}^2). \tag{20}$$

Note: Any two distributions with kernels $\tilde{G}_1(m,l) = \tilde{G}_2(m,-l)$ have the same variance, since

$$\sum\nolimits_l \tilde{G}_1(m_1,l)\tilde{G}_1^*(m_2,l) =$$

$$\sum\nolimits_l \tilde{G}_1(m_1,-l)\tilde{G}_1^*(m_2,-l)$$

$$= \sum\nolimits_l \tilde{G}_2(m_1,l)\tilde{G}_2^*(m_2,l).$$

Corollary: A distribution with real and symmetric product kernel $g(\nu\tau)$ and the distribution with its dual kernel $g_d(\nu\tau) = \mathcal{F}_{\alpha\to\nu,\beta\to\tau}[g(\alpha\beta)]$ have the same variance.

Proof: Consider all coordinates in the analog domain. The time-lag domain forms of $g(\nu\tau)$, $G(t,\tau) = \mathcal{F}_{\nu\to t}[g(\nu\tau)]$, and $g_d(\nu\tau)$, $G_d(\nu\tau) = \mathcal{F}_{\nu\to t}[g_d(\nu\tau)]$ are related by $G(t,\tau) = G_d(\tau,t)$. In the rotated domain this relation produces $\tilde{G}(t_1,t_2) = \tilde{G}_d(t_1,-t_2)$, what ends the proof, according to the previous note.

Example: The WD has the kernel $g(\nu\tau) = 1$, $\tilde{G}(m,l) = \delta(m+l)$. According to the Corollary, the WD has the same variance as its dual kernel counterpart, with $g(\nu\tau) = \delta(\nu,\tau)$, $\tilde{G}(m,l) = \delta(m-l)$. This dual kernel corresponds to the signal energy $\sum_m |x(n+m)|^2$ (see (16)). Thus, the WD and the signal energy have the same variance. The same holds for the smoothed spectrogram, and the S-method [10], [Article 6.2], whose kernels are $\tilde{G}(m,l) = w(m)p(m+l)w(l)$, and $\tilde{G}(m,l) = w(m)p(m-l)w(l)$, respectively. Their variance is the same.

Eigenvalue decomposition: Assume that both the summation limits and values of $\tilde{G}(m,l)$ are finite. It is true when the kernel $G(m,l)$ is calculated from the well defined kernel in a finite Doppler-lag domain, $G(m,l) = \mathcal{F}_{\nu\to m}[g(\nu,l)]$, using a finite number of samples. The signal dependent part of the variance $\sigma_{s\epsilon}^2(n,f)$ can be calculated, like other distributions from the Cohen class, by using eigenvalue decomposition of matrix $\tilde{\mathbf{G}}$, [2], [3]. The distribution of nonnoisy signal (16) is

$$\rho_s(n,f) = \sum_{i=-N/2}^{N/2-1} \lambda_i S_s(n,f;q_i) = \rho_s(n,f;\lambda,q), \tag{21}$$

where λ_i and $q_i(m)$ are eigenvalues and eigenvectors of the matrix $\tilde{\mathbf{G}}$, respectively, and $S_s(n,f;q_i) = \left|\sum\limits_{i=-N/2}^{N/2-1} s(n+m)q_i(m)e^{-j2\pi fm}\right|^2$ is the spectrogram of signal $s(n)$ calculated by using $q_i(m)$ as a lag window. Since $\tilde{\mathbf{\Phi}} = \sigma_\epsilon^2 \tilde{\mathbf{G}}^2$, its eigenvalues and eigenvectors are $\sigma_\epsilon^2 |\lambda_i|^2$ and $q_i(m)$, respectively. Thus, according to (20)

$$\sigma_{s\epsilon}^2(n,f) = 2\sigma_\epsilon^2 \sum_{i=-N/2}^{N/2-1} |\lambda_i|^2 S_s(n,f;q_i)$$

$$= 2\sigma_\epsilon^2 \rho_s(n,f;|\lambda|^2,q). \tag{22}$$

Relation between the original kernel and variance $\sigma_{s\epsilon}^2(n,f)$ kernel: According to (21), we can conclude that the original kernel in the Doppler-lag domain can be decomposed into $g(\nu,l) = \sum_{i=-N/2}^{N/2-1} \lambda_i a_i(\nu,l)$, where $a_i(\nu,l)$ are ambiguity functions of the eigenvectors $q_i(m)$. The kernel of $\rho_s(n,f;|\lambda|^2,q)$, in (22), is $g_\sigma(\nu,l) = \sum_{i=-N/2}^{N/2-1} |\lambda_i|^2 a_i(\nu,l)$. A detailed analysis of distributions, with respect to their eigenvalue properties, is presented in [3], [Article 6.4]. In the sense of that analysis, the signal dependent variance is just "an energetic map of the time-frequency distribution" of the original signal.

The mean value of variance (17) is:

$$\overline{\sigma_{s\epsilon}^2(n,f)} = \int_{-1/2}^{1/2} \sigma_{s\epsilon}^2(n,f)df$$

$$= 2\sigma_\epsilon^2 \sum\nolimits_m \tilde{\Phi}(m,m)\,|s(n+m)|^2. \tag{23}$$

For frequency modulated signals $s(n) = a\exp[j\phi(n)]$ it is a constant proportional to the kernel energy [1].

Special case 2: For **nonstationary white**

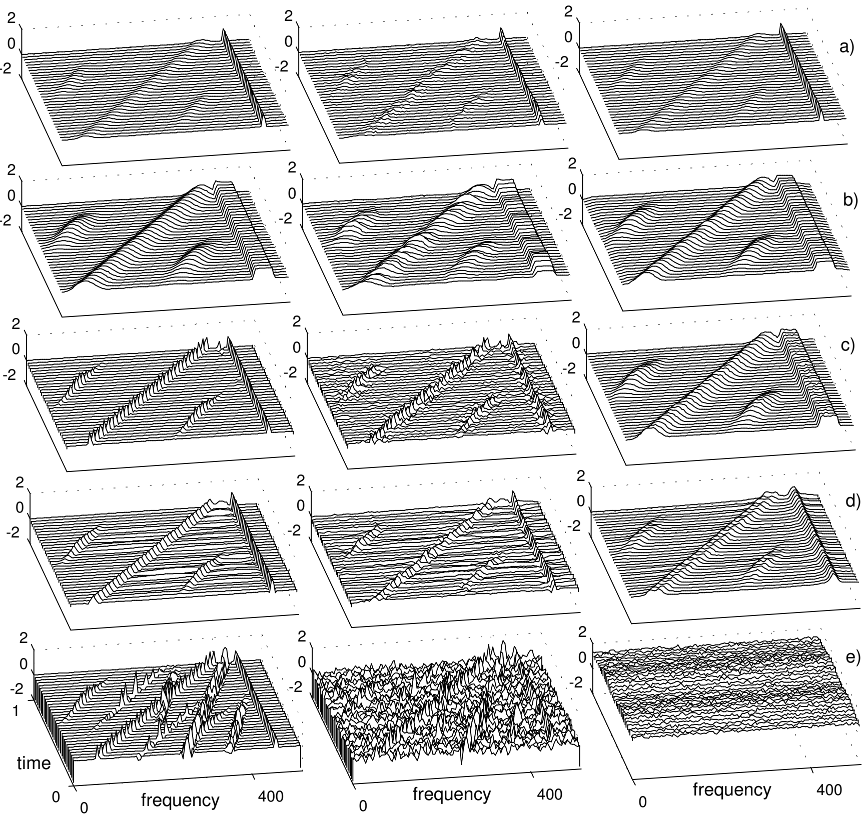

Fig. 1. Time-frequency representations of a nonnoisy signal (First column); One realization of time-frequency representations of the signal corrupted by a white stationary complex noise (Second column); Variances of the distributions, obtained numerically by averaging over 1000 realizations (Third column): a) Spectrogram, b) Smoothed spectrogram, c) S-method, d) Choi-Williams distribution, e) Pseudo Wigner distribution.

complex noise, (18) results in:

$$\tilde{\Phi}(m_1,m_2)=\sum_{i=-N/2}^{N/2-1} I(n+l)\tilde{G}(m_1,l)\tilde{G}^*(m_2,l), \tag{24}$$

or $\tilde{\mathbf{\Phi}} = \tilde{\mathbf{G}}\mathbf{I}_n\tilde{\mathbf{G}}$, where $\mathbf{I}_n$ is a diagonal matrix, with the elements $I(n+l)$. For the quasistationary case, $I(n+l_1)\delta(l_1-l_2) \cong I(n)\delta(l_1-l_2)$, we have $\mu_i = I(n)\left|\lambda_i\right|^2$, with all other parameters as in (22).

Special case 3: In the case of **colored stationary complex noise,** relations dual to those in Special case 2, hold (like (9) and (11)).

Special case 4: Let $x(n) = s(n)(1+\mu(n))$, where $\mu(n)$ is a **multiplicative noise.** We can write $x(n) = s(n)+s(n)\mu(n) = s(n)+\epsilon(n)$, where $\epsilon(n) = s(n)\mu(n)$ is an additive noise. Thus, the case of this kind of multiplicative noise can be analyzed in the same way as the additive noise. For example, if the noise $\mu(n)$ is a nonstationary white complex one with $R_{\mu\mu}(m,n) = I_\mu(n)\delta(n-m)$, then $R_{\epsilon\epsilon}(m,n) = I_\epsilon(n)\delta(n-m)$, where $I_\epsilon(n) = \left|s(n)\right|^2 I_\mu(n)$.

D. Numerical Example

Consider the signal

$$x(t) = \exp(j1100(t+0.1)^2)$$
$$+e^{-25(t-0.25)^2}\exp(j1000(t+0.75)^2)$$
$$+e^{-25(t-0.67)^2}\exp(j1000(t-0.4)^2)$$
$$+\exp(j2850t)+\epsilon(t),$$

within the interval $[0,1]$, sampled at $\Delta t = 1/1024$. A Hanning lag window of the width $T_w = 1/4$ is used. Stationary white complex noise with variance $\sigma_\epsilon^2 = 2$ is assumed. The spectrogram, smoothed spectrogram, S-method, Choi-Williams distribution (CWD), and the WD, of signal without noise are presented in the first column of Fig. 1, respectively. For the CWD, the kernel $g(\nu,\tau) = \exp(-(\nu\tau)^2)$ is used, with normalized coordinates $-\sqrt{\pi N/2} \le |2\pi\nu| < \sqrt{\pi N/2}$, $-\sqrt{\pi N/2} \le |\tau| < \sqrt{\pi N/2}$, and 128 samples within the intervals. Elements of the matrix $\tilde{\mathbf{G}}$ were calculated as, [3]

$$\tilde{G}(m,l) = \sum_{p=-N/2}^{N/2} g(p\Delta\nu,(m-l)\Delta\tau) \times \exp(-j2\pi(m+l)p/(2N))\Delta\nu. \quad (25)$$

The normalized eigenvalues of the matrix $\tilde{\mathbf{\Phi}}$ were $\lambda_i = \{1,-0.87,0.69,-0.58,0.41,-0.30,...\}$ and $\mu_i = |\lambda_i|^2 = \{1,0.76,0.47,0.33,0.17,0.09,...\}$. In the spectrogram and smoothed spectrogram the whole signal dependent part of variance is "located" just on the signal components, while in the WD it is "spread" over the entire time-frequency plane. Variance behavior in other two distributions is between these two extreme cases. As it has been shown, the variances in the smoothed spectrogram and the S-method are the same (Fig. 1(b) and (c)).

E. Summary

The variance values for a white nonstationary complex noise, with $R_{\epsilon\epsilon}(m,n) = I(n)\delta(m-n)$, $I(n) \ge 0$, for some distributions, are summarized next.

-Pseudo Wigner distribution $W_s(n,f;w)$, with $\tilde{G}(m,l) = w(m)\delta(m+l)w(l)$:

$$\sigma_{WD}^2(n,f) = \sigma_{\epsilon\epsilon}^2(n,f) + \sigma_{s\epsilon}^2(n,f) = W_I(n,0;w^2) + 2W_{I,|s|^2}(n,0;w^2), \quad (26)$$

where $W_{I,|s|^2}$ denotes the cross Wigner distribution for $I(n)$ and $|s(n)|^2$.

-Spectrogram $S_s(n,f;w)$, with $\tilde{G}(m,l) = w(m)w(l)$:

$$\sigma_{SPEC}^2(n,f) = S_I(n,0;w^2) +2F_I(n,0;w^2)S_s(n,f;w). \quad (27)$$

The STFT of $I(n)$, calculated using the window $w^2(m)$, is denoted by $F_I(n,f;w^2)$.

-A general quadratic distribution, with kernel $\tilde{G}(m,l) = G((m+l)/2,(m-l)/2)$, in (6) or (16), and $\tilde{\mathbf{G}}$ being a matrix with elements $\tilde{G}(m,l)$:

$$\sigma_\rho^2(n,f) = \rho_I(n,0;|G|^2) + 2\rho_s(n,f;\tilde{\mathbf{G}}\mathbf{I}_n\tilde{\mathbf{G}}). \quad (28)$$

First two formulae are special cases of (28). Expressions for stationary white noise follow with $I(n) = \sigma_\epsilon^2$. Dual expressions hold for a colored stationary noise.

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