Instantaneous Frequency Estimation by Using the Wigner Distribution and Linear Interpolation

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Abstract— **Nonparametric algorithm for the instantaneous frequency (IF) estimation, by using the Wigner distribution (WD) with an adaptive window length, is considered in the paper. This algorithm produces a bias-to-variance trade-off close to optimal, meaning almost minimal mean squared error (MSE) of the estimation. Thus, the adaptive window length is characterized by a small bias at the considered instant. Then, according to the WD concentration property, the IF estimate can be assumed as a linear function within this window. Instead of nonparametric IF estimation in other points within this interval the linear IF interpolation can be performed. Length of the interpolation segment is determined based on the adaptive window length. It is done in such a way to produce a trade-off between the interpolation caused error and calculational complexity. This modification can produce a significant calculation savings, without increasing the overall MSE. Theoretical analysis has been confirmed on numerical examples and statistical study with four synthetic signals. The approach presented here can be generalized in a straightforward manner to nonlinear interpolations and higher order time-frequency representations.**

I. Introduction

Accurate instantaneous frequency (IF) estimation is an important problem in numerous applications [1]-[10]. Here we will focus our attention to the nonparametric IF estimation based on the time-frequency (TF) representations, in particular on the Wigner distribution (WD). The WD, like many other TF distributions, exhibits the property that its first moment along the frequency is equal to the IF [11]. This property holds for the nonnoisy signals. It can be used for nonparametric IF estimation only in low to moderate noise environments [12]. The WD is concentrated around signals' IF [1]. This fact has motivated introduction of the IF estimator based on positions of the WD maxima [1]-[3]. This estimator is a simple and commonly used nonparametric estimation tool. Statistical performance of this estimator is analyzed in [1]-[3]. It has been shown that the bias, caused by the IF non-linearity, is proportional to a power of the lag window length, while the variance, caused by the noise, is a decreasing function of the lag window length. Thus, the bias-to-variance trade-off, producing the minimal mean squared error (MSE), exists.

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The nonparametric algorithm for calculation of the adaptive window length in the WD is proposed in [3]. It is based on the rule of confidence intervals intersection. Algorithm sources are in the nonparametric regression. Recently this algorithm is applied to other TF representations, including the Polynomial Wigner-Ville distribution (PWVD) [13]. The main algorithm drawback is in calculation of several WDs at each time instant.

In this paper we propose a modification of the nonparametric algorithm. The algorithm results in a window length such that the bias is relatively small (of the same order of magnitude as the estimate's variance) within the considered lag interval. Small bias for the WD based estimator means that the IF can be considered as a linear function within the lag window. Thus, in calculation we can skip an interval proportional to the adaptive lag window length. Within this interval the IF estimate is interpolated with a linear function. Furthermore, the interpolation means that we can skip WD calculations in the other points within this interval. It results in a reduction of calculation complexity, since the number of points where the nonparametric algorithm is performed (with calculation of several WDs) is decreased. Deviation of the true IF from the linear behavior could increase the MSE. Therefore, the interval length should be chosen as a trade-off between the calculation complexity and possible non-linearity error.

The paper is organized as follows. Performance of the WD based IF estimator is reviewed within Section II. This section ends with a description of the nonparametric algorithm for the IF estimation, which is based on the intersection of confidence intervals rule. The proposed algorithm modification is introduced and presented in Section III. Comparison of the calculation complexity between the original and the modified algorithm is given. Theoretical analysis is confirmed in Section IV, where numerical examples and statistical study are done.

II. The WD as an IF Estimator

An important property of TF representations is that they are concentrated around the signals' IF [1]-[10]. This is the basis for a group of methods for the IF estimation, which use detection of the TF representations maxima positions. For FM signals, $f(t) = Ae^{j\phi(t)}$, the IF is defined as the first derivative of the phase, $\omega(t) = \phi'(t)$. Based on a TF representation $TF(t,\omega)$, the IF is estimated as:

$$\hat{\omega}(t) = \arg\max_{\omega} TF(t,\omega). \tag{1}$$

Here, a special attention will be paid to the WD:

$$WD_h(t,\omega) = \sum_k w_h(kT)x(t+kT)x^*(t-kT)e^{-j2\omega kT}, \tag{2}$$

of a noisy signal

$$x(t) = f(t) + \nu(t), \tag{3}$$

where T is the sampling interval, $\nu(t)$ is a white, Gaussian noise with independent real and imaginary part with variance $2\sigma^2$, and $w_h(t)$ is a lag window whose length is h, i.e., $w_h(t) = 0$ for $|t| \geq h/2$. The WD of signal (3) can be written in the form:

$$WD_h(t,\omega) = A^2 \times \sum_k w_h(kT)e^{j\phi(t+kT)-j\phi(t-kT)}e^{-j2\omega kT} + WD_\nu(t,\omega), \tag{4}$$

where $WD_\nu(t,\omega)$ denotes the noise-term, given by

$$WD_\nu(t,\omega) = \sum_k w_h(kT)[f(t+kT)\nu^*(t-kT) + \nu(t+kT)f^*(t-kT) + \nu(t+kT)\nu^*(t-kT)]e^{-j2\omega kT}. \tag{5}$$

The signal phase can be expanded into a Taylor series up to the third order term

$$\phi(t+kT) - \phi(t-kT) \approx 2\phi'(t)(kT) + \frac{2\phi'''(t)(kT)^3}{3!}. \tag{6}$$

For small $\phi'''(t)(nT)^3/3!$ we can make the approximation: $\exp(j2\phi'''(t)(kT)^3/3!) \approx 1 + j2\phi'''(t)(kT)^3/3!$, so that expression (4) can be written as:

$$\begin{aligned} WD_h(t,\omega) &= A^2 \sum_k w_h(kT)e^{j2\phi'(t)kT}e^{-j2\omega kT} \\ &+ A^2 \sum_k j\frac{1}{3}\phi'''(t)(kT)^3 w_h(kT)e^{j2\phi'(t)kT}e^{-j2\omega kT} \\ &+ WD_\nu(t,\omega) \\ &= WD_1(t,\omega) + WD_3(t,\omega) + WD_\nu(t,\omega). \end{aligned} \tag{7}$$

Note that the first term $WD_1(t,\omega)$ has maxima along the IF $\omega(t) = \phi'(t)$, where $\partial WD_1(t,\omega)/\partial\omega|_{\omega=\phi'(t)} = 0$. Other two terms in (7), $WD_3(t,\omega)$ and $WD_\nu(t,\omega)$, produce the IF estimation error. A linearization of $\partial WD_h(t,\omega)/\partial\omega$, around the exact IF value $\omega(t) = \phi'(t)$, is performed in [3], in order to calculate the errors which appear in this estimator. It is assumed that the errors caused by $WD_3(t,\omega)$ and $WD_\nu(t,\omega)$ are statistically independent. Then we can write:

$$\begin{aligned} \frac{\partial WD_h(t,\omega)}{\partial\omega}|_0 &= \\ \frac{\partial WD_1(t,\omega)}{\partial\omega}|_0 &+ \frac{\partial^2 WD_1(t,\omega)}{\partial\omega^2}|_0 \Delta\omega(t) \\ &+ \frac{\partial WD_3(t,\omega)}{\partial\omega}|_0 + \frac{\partial WD_\nu(t,\omega)}{\partial\omega}|_0 = 0, \end{aligned} \tag{8}$$

where $|_0$ denotes the point $\omega = \phi'(t)$. From (8) we get an expression for the IF estimation error $\Delta\omega(t)$:

$$\Delta\omega(t) = \hat{\omega}(t) - \omega(t)$$
$$= -\frac{\frac{\partial WD_3(t,\omega)}{\partial\omega}|_0 + \frac{\partial WD_\nu(t,\omega)}{\partial\omega}|_0}{\frac{\partial^2 WD_1(t,\omega)}{\partial\omega^2}|_0}. \tag{9}$$

Expected value of the IF estimation error is, [3], [14]:

$$E\{\Delta\omega(t)\} = -\frac{\frac{\partial WD_3(t,\omega)}{\partial\omega}|_0}{\frac{\partial^2 WD_1(t,\omega)}{\partial\omega^2}|_0} =$$
$$\frac{-\sum_k \frac{2(kT)}{3}\phi'''(t)(kT)^3 w_h(kT)e^{j2\phi'(t)kT-j2\phi'(t)kT}}{\sum_k(-2jkT)^2 w_h(kT)e^{j2\phi'(t)kT-j2\phi'(t)kT}}$$
$$= k_b\phi'''(t)h^2 = k_b\omega^{(2)}(t)h^2, \tag{10}$$

while the variance is given by:

$$var\{\Delta\omega(t)\} = \frac{E\left\{\left|\frac{\partial WD_\nu(t,\omega)}{\partial\omega}|_0\right|^2\right\}}{\left|\frac{\partial^2 WD_1(t,\omega)}{\partial\omega^2}|_0\right|^2}$$
$$= \frac{\sigma^2}{A^2}\left(1+\frac{\sigma^2}{A^2}\right)\frac{\sum_k(kT)^2 w_h^2(kT)}{(\sum_k(kT)^2 w_h(kT))^2}$$
$$= \frac{\sigma^2}{A^2}\left(1+\frac{\sigma^2}{A^2}\right)k_v h^{-3}. \tag{11}$$

The constants k_b and k_v depend on the window shape. For common lag windows they are given in Table I. Expressions for k_b and k_v directly follow from the relations (10) and (11), [3], [14]. The estimator's variance (11) is a decreasing function, while the bias (10) is an increasing function of the window length h. The MSE of the IF estimation error is:

$$MSE = \left(k_b\omega^{(2)}(t)h^2\right)^2 + \frac{\sigma^2}{A^2}\left(1+\frac{\sigma^2}{A^2}\right)k_v h^{-3}. \tag{12}$$

From (12) we can find the optimal window length h. By minimizing the MSE, $\partial MSE/\partial h|_{h=\hat{h}(t)} = 0$, we get:

$$\hat{h}(t) = \sqrt[7]{\frac{3\frac{\sigma^2}{A^2}\left(1+\frac{\sigma^2}{A^2}\right)k_v}{4k_b^2[\omega^{(2)}(t)]^2}}. \tag{13}$$

This expression cannot be used in practical realizations since it contains unknown derivative of the IF $\omega^{(2)}(t)$. However, it was used for development of the adaptive algorithm for the IF estimation, based on the confidence intervals intersection [3], [14]. This algorithm can produce solutions close to the optimal ones. Note that for the optimal window length the equality $E\{\Delta\omega(t)\} = \sqrt{\frac{3}{4}var\{\Delta\omega(t)\}}$ holds.

In order to illustrate the above relations and errors in the IF estimation we will present an example.

Example: Consider the signal $f(t) = \exp(j\phi(t))$, with the IF given by $\omega(t) = \phi'(t) = 24\mathrm{atan}(16t)$, within the time interval $t \in [-1,1]$, with the sampling interval $T = 1/512$. The signal is embedded in a Gaussian noise with the standard deviation of imaginary and real part having the value of $\sigma = 0.15$. The IF estimates, based on the maxima of the WD with a rectangular window of the length $h = NT$, for the cases of $N = 512$, $N = 128$, and $N = 8$, are shown in Figures 1a,b,c, respectively. It can be seen that wider window lengths produce larger bias and smaller variance, while smaller window lengths produce smaller bias and larger variance, according to (12). The MSE error of IF estimation, determined by using Monte Carlo simulations, as a function of the window length, is shown in Figure 1d for three different amounts of noise $\sigma = 0.1$, $\sigma = 0.3$ and $\sigma = 0.7$. For the highest amount of noise the MSE minimum is reached for a wide window length, since the influence of noise on the IF estimates is the smallest for that window length. The WDs with $N = 512$ and $N = 8$, corresponding to estimates in Figures 1a,c, are shown in Figures 1e,f, respectively.

A. *Specific Statistical Approach Based on Confidence Intervals Intersection*

A specific statistical approach for the IF estimation is proposed in [3]. Its origin is in nonparametric regression. An adaptive, close to the optimal, IF estimate is obtained by using adaptive time-varying window lengths [3]. The adaptive window length $\hat{h}(t)$ is chosen, at each time instant, from a set of the windows

TABLE I
PARAMETERS k_v AND k_b FOR COMMON WINDOW TYPES.

Window type	Rectangular	Hanning	Triangular	Hamming
k_v	0.0833	0.0163	0.0208	0.0217
k_b	0.0250	0.0131	0.0167	0.0168

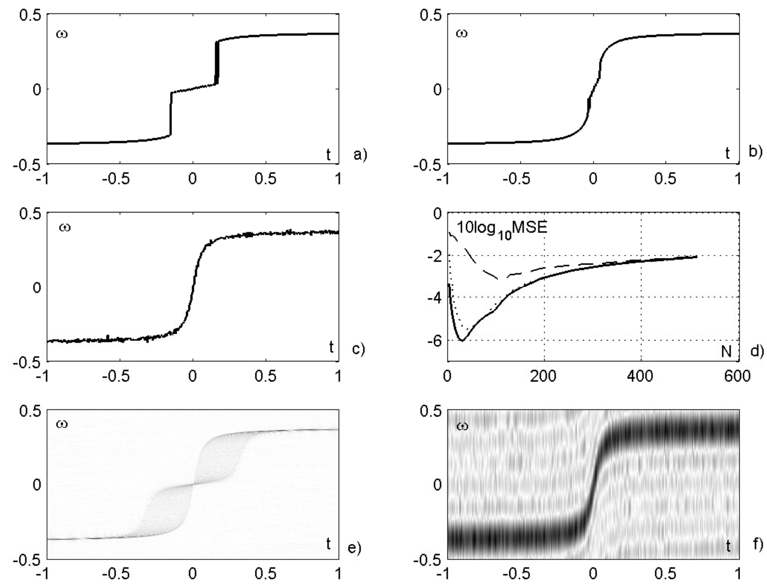

Fig. 1. Instantaneous frequency estimation based on the WD maxima: a) $N = 512$; b) $N = 128$; c) $N = 8$; d) MSE ($\sigma = 0.1$ - solid line; $\sigma = 0.3$ - dotted line; $\sigma = 0.7$ - dashed line); e) WD with $N = 512$; f) WD with $N = 8$.

with dyadic lengths

$$\mathbf{H} = \{h_k = h_0 2^k, k = 0, 1, 2, ..., J\}, \qquad (14)$$

where h_0 is an initial, narrow window. Estimate of the optimal window $\hat{h}(t)$ is determined by using the rule of confidence intervals intersection, producing a bias-to-variance trade-off [3]. Here, we will briefly explain its basic idea.

The IF estimates are obtained based on the WD with various window lengths from the set $\mathbf{H}$:

$$\hat{\omega}_{h_k}(t) = \underset{\omega}{\arg\max} WD_{h_k}(t, \omega), \quad k = 0, 1, 2, ..., J. \qquad (15)$$

The confidence intervals for each estimate are defined as:

$$D_{h_k} = [\hat{\omega}_{h_k}(t) - \kappa\sigma(h_k), \hat{\omega}_{h_k}(t) + \kappa\sigma(h_k)], \quad (16)$$

where κ determines probability that the mean value of $\hat{\omega}_{h_k}(t)$ is within the confidence interval, while $\sigma(h_k)$ is the standard deviation of the IF estimate (11). For example, for Gaussian distribution of random variable $\hat{\omega}_{h_k}(t)$, $\kappa = 3$ means that the mean value is within D_{h_k} with probability of 0.997. When the bias is so small that it can be neglected, the mean value of $\hat{\omega}_{h_k}(t)$ is equal to the true IF $\omega(t)$. Thus, the confidence intervals intersect when the bias is small, since they contain the true IF value. When we increase the window length the bias is increased, while the variance is decreased. For large window lengths the bias becomes so large that the mean values of $\hat{\omega}_{h_k}(t)$ have large difference, due to the bias, and the consecutive confidence intervals stop to intersect. The specific statistical approach proposes comparison of the confidence intervals of estimates produced with consecutive window lengths from the set $\mathbf{H}$. In order to take into account a small bias, up to $E\{\Delta\omega(t)\} \approx \sqrt{var\{\Delta\omega(t)\}}$, one should modify the confidence intervals as:

$$D_{h_k} = [\hat{\omega}_{h_k}(t) - (\kappa+1)\sigma(h_k), \hat{\omega}_{h_k}(t) + (\kappa+1)\sigma(h_k)]. \quad (17)$$

The adaptive window length, with an appropriate bias-to-variance trade-off, is chosen as the widest one from the set $\mathbf{H}$ when two consecutive intervals still intersect.

Algorithm

The algorithm for the IF estimation can be summarized as follows:

1) Consider an instant $t = t_0$.

2) Calculate the WD with the narrowest window from the set $h_0 \in \mathbf{H}$, $WD_{h_0}(t,\omega)$, $k = 0$.

3) Initial IF estimate is:

$$\hat{\omega}_{h_0}(t) = \arg\max_{\omega} WD_{h_0}(t,\omega). \quad (18)$$

4) Consider next window length $k = k + 1, h_k = h_0 2^k$, calculate the WD for this window $WD_{h_k}(t,\omega)$, and determine corresponding IF estimate:

$$\hat{\omega}_{h_k}(t) = \arg\max_{\omega} WD_{h_k}(t,\omega). \quad (19)$$

5) If the inequality:

$$|\hat{\omega}_{h_k}(t) - \hat{\omega}_{h_{k-1}}(t)| \leq (\kappa+1)[\hat{\sigma}(h_k) + \hat{\sigma}(h_{k-1})], \quad (20)$$

is not satisfied or h_k is the widest window from the set $\mathbf{H}$, then data-driven window is $\hat{h}(t) = h_{k-1}(t)$, and the adaptive IF estimate is $\hat{\omega}(t) = \hat{\omega}_{h_{k-1}}(t)$. Otherwise, go to step 4.

6) $t = t + T$, go to step 1.

Value κ in (20) is set to 3. Procedure for the standard deviation estimation (20) can be performed at the beginning of the algorithm, based on the formula given in [3], [14]. Values of $(\kappa+1)[\hat{\sigma}(h_k) + \hat{\sigma}(h_{k-1})]$ in (20) can be stored in a look up table. The same algorithm can be used with other time-frequency representations. Its implementation in the case of the PWVD is described in [13].

III. Proposed Algorithm

The proposed algorithm is a modification of the previously described nonparametric algorithm. This modification is proposed in order to reduce the calculation complexity. In the adaptive algorithm the estimation starts from the narrowest window, producing the smallest bias and the largest variance. Then, twice wider window is used in each successive estimation for the considered time instant. If the confidence intervals of two consecutive windows intersect, it means that the bias is still small, as compared to the variance. Note that from (10) follows that the bias is proportional to the squared IF's second derivative. Thus, from the same relation follows that small bias means small IF's second derivative, i.e., the IF, estimated by the WD, can be treated as a linear function within the considered window. We can conclude that the IF can be represented by a straight line within an interval determined by the optimal window. In this way, by using the adaptive statistical approach we get the IF estimation, as well as the interval where we can use parametric IF estimation, with a satisfactory overall precision.

Consider a time interval $[t_1, t_2]$. The IF estimates at the end points, $\hat{\omega}(t_i)$, $i = 1, 2$, are obtained by using the presented statistical method. Within the interval (t_1, t_2), the IF is interpolated by a line that contains points $(t_i, \hat{\omega}(t_i))$, $i = 1, 2$. Let us now analyze the influence of interpolation on the IF estimation accuracy. Neglecting other sources of errors, assume that the exact IF values at the points $(t_i, \omega(t_i))$, $i = 1, 2$, are obtained. Let the IF be a continuous function with continuous and limited derivatives within the interval (t_1, t_2). The IF within the interval is interpolated with a line:

$$\hat{\omega}(t) = a\omega(t_2) + (1-a)\omega(t_1), \quad (21)$$

where $a = (t - t_1)/(t_2 - t_1)$, and $t \in [t_1, t_2]$ is an arbitrary point. Expanding the functions $\omega(t_i)$, $i = 1, 2$, into the Taylor series around a point $t \in [t_1, t_2]$, up to the second order term, we get:

$$\hat{\omega}(t) = \omega(t) - a(1-a)\omega^{(2)}(t)(\Delta t)^2/2, \quad (22)$$

where $\Delta t = t_2 - t_1$ is the interval length. From (22) we can conclude:

-If $\Delta t = t_2 - t_1$ is of the window length order of magnitude, then the interpolation error behaves in the same way as the estimation bias in the considered point.

-The largest error, with respect to the variation of parameter a only, occurs in the middle of the considered interval, $a = 0.5$. It can be written as:

$$e_{app}(t) = -\omega^{(2)}(t)(\Delta t)^2/8. \qquad (23)$$

We want to perform a linear interpolation of the IF values in such a way that the calculation savings (i.e., wide interpolation intervals) do not cause a significant increase of the MSE. To this aim we decided to keep the error $e_{pp}(t)$, caused by the introduced interpolation, less than a portion of the IF estimation bias $E\{\Delta\omega(t)\}$ value:

$$\left|\frac{e_{pp}(t)}{E\{\Delta\omega(t)\}}\right| = \left|\frac{\omega^{(2)}(t)(\Delta t)^2/8}{k_b\omega^{(2)}(t)h^2}\right| \leq \eta. \qquad (24)$$

>From (24) follows $\Delta t \leq h\sqrt{8\eta k_b}$. For example, for the rectangular window $k_b = 1/40$ and maximal error introduced by the interpolation equal to 10% of the bias, we have $\eta = 0.1$, $\Delta t \leq h/5\sqrt{2}$. We have set the interpolation interval length to the first smaller value that corresponds to a window length from the set (14). It is $\Delta t = h/8$ and it will produce an integer number of skipped instants. With $\Delta t = h/4$ maximal interpolation error is 31.2% of the bias. This is the reason why, in the proposed algorithm modification, we have used the interpolation interval length equal to $\hat{h}(t)/8$, where $\hat{h}(t)$ is the adaptive window length. Narrower interpolation intervals would produce small improvements of accuracy, but they would reduce calculation savings. Wider interpolation intervals would mean a significant influence of the interpolation error. For rectangular window the step size of $\hat{h}(t)/8$ produces an error of 7.8% of the bias $|e_{app}(t)| = 0.078E\{\Delta\omega(t)\}$, while for other windows from Table I this step will produce an interpolation error slightly higher than 10%.

Therefore, *we can use linear interpolation, without a significant increase of the estimation error, within the interval $s = \hat{h}(t)/8$, where $\hat{h}(t)$ is the optimal window length of the estimator in the considered point.*

Algorithm:

1. Consider instant t_0, $i = 0$.

2. Determine the IF estimate $\hat{\omega}(t_i)$ and adaptive window $\hat{h}(t_i)$ by using the presented statistical approach. If $i \neq 0$ then interpolate the IF within the interval $t \in [t_{i-1}, t_i]$ by using (21).

3. Next time instant is $t_{i+1} = t_i + s_i$, where $s_i = \hat{h}(t_i)/8$ is the interval for parametric interpolation. Go to 2.

Note: In the presented analysis we have used only the dominant term in the bias and in the approximation error. It is proportional to the second IF derivative $\omega^{(2)}(t)$. In general, the bias depends on all even order derivatives, $E\{\Delta\omega(t)\} = \sum_{l=1}^{\infty} k_l\omega^{(2l)}(t)h^{2l}$. The approximation error contains all IF derivatives, $e_{app} = \sum_{l=2}^{\infty} q_l\omega^{(l)}(t)(\Delta t)^l$. At some points it can happen that all $\omega^{(2l)}(t)$ are very small, meaning small bias, while $\omega^{(2l+1)}(t)$ are large, meaning large approximation error. From analysis that we performed for this case we have concluded that, in order to treat the problems related to this effect, we should not allow the step size to increase more than twice between two consecutive points. In the presented algorithm it means: if $\hat{h}(t_i)/8 > 2s_{i-1}$ then $s_i = 2s_{i-1}$. This kind of problem can also be reduced by using higher order TF representations, like for example PWVD.

A. Calculation Complexity

The calculation complexity of the WD with window length N is $N(3 + \log_2 N)/4$ complex multiplications, and $(N\log_2 N)/2$ complex additions [15]. By assuming that the complex multiplication can be obtained by using 4 real multiplications, and two real additions, as well as that complex addition represents two real additions, we can conclude that the calculation of the WD takes $N(3 + \log_2 N)$ real multiplications, and $3N(1 + \log_2 N)/2$ real additions. If the adaptive window in the considered point is $\hat{h}(t) = 2^K h(t_0)$, then application of the presented statistical approach requires calculation of $K + 1$ WDs. In order to avoid error caused by the discrete nature of frequency, zero-padding to the widest window from the set $\mathbf{H}$ is done [3]. In addition, the IF estimation by using position of maxima needs $N-1$ comparisons, while (20) needs additional K comparisons.

Thus, overall calculation complexity of the algorithm based on the confidence intervals intersection is:

$$(K+1)N(3+\log_2 N) \text{ real multiplications,}$$

$$(K+1)3N(1+\log_2 N)/2 \text{ real additions,}$$

$$\text{and } (K+1)(N-1)+K \text{ comparisons,} \quad (25)$$

for a considered point. The proposed approach has the same calculation complexity for the instant where the IF estimation is performed by this statistical approach. However, this approach results in skipping of the next $2^K h(t_0)/8T$ points, since the interpolation is done in $(2^K h(t_0)/8T-1)$ points. Interpolation of the IF needs 4 real additions, 2 real multiplications and 1 real division for each instant. An average calculation complexity of the algorithm for that interval is:

$$\frac{(K+1)N(3+\log_2 N)+2(2^K h(t_0)/8T-1)}{2^K h(t_0)/8T}$$

real multiplications,

$$\frac{(K+1)3N(1+\log_2 N)/2+4(2^K h(t_0)/8T-1)}{2^K h(t_0)/8T}$$

real additions

$$\frac{2^K h(t_0)/8T-1}{2^K h(t_0)/8T} \text{ real divisions,}$$

$$\text{and } \frac{(K+1)(N-1)+K}{2^K h(t_0)/8T} \text{ comparisons.} \quad (26)$$

Assume, for example, the case when the widest window length is $N=512$ samples, and the narrowest window length is $N=4$, as well as that the optimal window length for a considered instant is $N=32$. This optimal window length means that we can skip next $32/8=4$ instants, and use the parametric model within this interval of 4 instants. By using the proposed algorithm we will get approximately 4 times lower calculation complexity than by using the presented statistical approach at each instant. For the assumed values, according to (25) and (26), the IF estimation in the considered instant, needs:

-22528 $\times$ 4 real multiplications, 27684 $\times$ 4 real additions, and 2047 $\times$ 4 comparisons, for a direct application of the specific statistical approach in these 4 instants, or

-22528 + 6 real multiplications, 27684 + 12 real additions, 2047 comparisons, and 3 real divisions for performing the proposed algorithm for the IF estimation in this interval of 4 instants.

Of course, in real data analysis the calculation complexity depends on the signal and noise characteristics. It can be analyzed only statistically (Section IV). Note that the procedure for the WDs calculation (with different window lengths) can be parallelized. Thus, in multiprocessor systems all WDs can be calculated in parallel for the same amount of time. Also, calculation of the WD with a narrow window contains a large number of zero values. It can be simplified in the first latices of the FFT algorithm. This fact can be used for a reduction of calculation complexity in monoprocessor systems.

IV. Example

In order to illustrate the algorithm we have considered the WD based IF estimator and four signals with: $\omega_1(t)=a_1 t$, $\omega_2(t)=a_2 t^2$, $\omega_3(t)=a_3|t|$, and $\omega_4(t)=a_4 \mathrm{sgn}(t)$. Parameter values are $(a_1,a_2,a_3,a_4)=(256\pi,576\pi,256\pi,128\pi)$. In all examples the SNR is $20\log_{10}(A/\sigma)=15[dB]$. The considered interval is $t\in[-0.5,0.5]$, with $N_t=512$ samples. Since the error caused by a discrete frequency grid can be significant, zero-padding to the widest window length $N=512$ has been done. The IF estimates are shown in Fig.2, as well as the adaptive window lengths produced by the statistical approach. The mean absolute IF estimation error for the WD: a) with a constant window length from the set $\mathbf{H}$ producing minimal mean absolute error (which is not known in advance); b) with the adaptive algorithm for each point, and c) for the adaptive algorithm

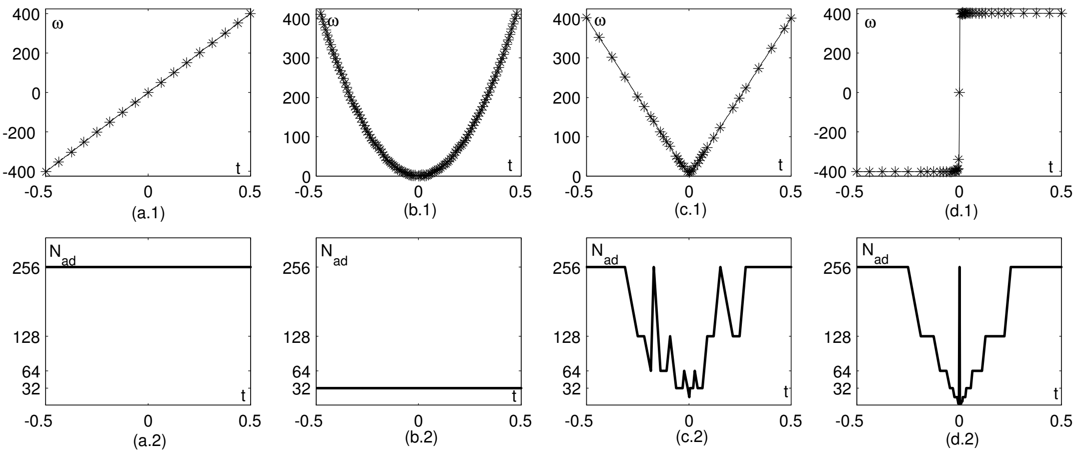

Fig. 2. The IF estimates (first row), and adaptive window length (second row): a) $\omega_1(t)$, b) $\omega_2(t)$, c) $\omega_3(t)$, d) $\omega_4(t)$. Asterisks denote points where the nonparametric IF estimation by using the WD is done. The interpolation is performed between these points.

TABLE II

IF estimation by using the WD. N_c - the constant window width which produces the minimal mean absolute error. Indexes denote: c - estimation with N_c; a - estimation with the adaptive algorithm; i - estimation with the proposed algorithm. K_{mean} - average number of points where the IF estimation has been done. STAT - average number of the WD calculations for the specific statistical approach; PROP - average number of the WD calculations for the proposed algorithm.

Estimator	**Wigner distribution**			
Signal	$\omega_1(t)$	$\omega_2(t)$	$\omega_3(t)$	$\omega_4(t)$
N_c	256	32	32	8
$E\{\lvert\Delta\hat{\omega}_c\rvert\}$	0.08	1.28	1.29	11.02
$E\{\lvert\Delta\hat{\omega}_a\rvert\}$	0.08	1.28	0.99	0.80
$E\{\lvert\Delta\hat{\omega}_i\rvert\}$	0.15	1.15	0.61	1.13
N_t/K_{mean}	30.12	3.97	13.20	9.88
$STAT$	4096	2560	3654	3756
$PROP$	264	645	327	422

with interpolation, proposed here, is shown in rows 4, 5 and 6 of Table I, respectively. In all examples 25 trials are performed. Also, the ratio between number of samples where the IF is calculated by the standard algorithm and the average number of points for the proposed algorithm, N_t/K_{mean}, is given. An average number of the WDs calculated for the IF estimation by the specific statistical approach (STAT) and by the proposed algorithm (PROP) is also given in Table II. A decrease in calculation complexity by using the proposed algorithm is evident (Table II, last row).

V. Conclusion

The method for linear interpolation of the estimated IF values, based on the WD, is introduced. The interpolation interval is determined by using the optimal lag window length at the considered instant. In this way, an accurate and numerically efficient IF estimation algorithm is obtained. The method can be generalized in a straightforward manner for the application on the PWVD and other higher order TF representations by using polynomial interpolation forms.

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