

Robust DFT with high breakdown point for complex-valued impulse noise environment

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Abstract— Modification of the robust DFT is proposed in order to achieve high breakdown point for signals corrupted by complex-valued impulse noise with independent real and imaginary parts. Obtained results are compared with existing robust DFT forms. In addition, an adaptive procedure for selection of the modified robust DFT form is developed.

I. INTRODUCTION

In practice, signals are quite often corrupted by non-Gaussian or impulse noise. Such situations can result from target glint in radar signal processing [1], coding/decoding errors in data transmission, apparatus malfunction, atmospheric phenomena and man-made activities in communications [2], etc. Conventional spectral analysis techniques are inefficient in such cases. Recently, robust DFT forms have been proposed for spectral analysis of signals corrupted by impulse noise [2]:

$$x(n) = f(n) + \nu(n), \quad n \in [0, N), \quad (1)$$

where $f(n)$ is signal of interest, while $\nu(n)$ is a white impulse noise, and N is number of signal samples. We will consider here the L-filter form of the DFT (L-DFT) given as [3]:

$$X_L(\omega) = \sum_{m=0}^{N-1} a_m [\mathbf{r}_m(\omega) + \mathbf{j}\mathbf{i}_m(\omega)] \quad (2)$$

where coefficients a_m satisfy $\sum_{m=0}^{N-1} a_m = 1$ and $a_m = a_{N-1-m}$ for $m \in [0, N)$, while $\mathbf{r}_m(\omega)$ and $\mathbf{i}_m(\omega)$ are elements from the sets $\mathbf{R}(\omega)$ and $\mathbf{I}(\omega)$

$$\mathbf{R}(\omega) = \{\text{Re}\{\mathbf{x}(\mathbf{n}) \exp(-\mathbf{j}\omega\mathbf{n})\} \text{ for } \mathbf{n} \in [0, \mathbf{N}]\}$$

$$\mathbf{I}(\omega) = \{\text{Im}\{\mathbf{x}(\mathbf{n}) \exp(-\mathbf{j}\omega\mathbf{n})\} \text{ for } \mathbf{n} \in [0, \mathbf{N}]\}. \quad (3)$$

Values $\mathbf{r}_m(\omega)$ and $\mathbf{i}_m(\omega)$ are sorted into non-decreasing sequences: $\mathbf{r}_m(\omega) \leq \mathbf{r}_{m+1}(\omega)$ and $\mathbf{i}_m(\omega) \leq \mathbf{i}_{m+1}(\omega)$. The α -trimmed form of coefficients is commonly used (here given for even number of samples N): $a_m = 1/[2\alpha(N-2)+2]$ for $m \in [N/2-1-\alpha(N-2), N/2+\alpha(N-2)]$ and $a_m = 0$ elsewhere, where $\alpha \in [0, 1/2]$. Two special cases of the α -trimmed mean are:

- the standard DFT for $\alpha = 1/2$:

$$X_S(\omega) = \frac{1}{N} \sum_{n=0}^{N-1} x(n) \exp(-j\omega n). \quad (4)$$

- the median-filter DFT form for $\alpha = 0$.

These two transforms have quite different behavior. The standard DFT is very sensitive to impulses, while median-filter form, robust to impulse noise, exhibits spectral distortion effect. Taking this into account, it can be expected that there exists a trade-off in selection of α parameter. A general rule is that impulse rejection property of the α -trimmed mean filter improves with the decrease of α . At the same time, the spectral distortion effect becomes more considerable. Then, quasi-optimal parameter α , for the considered noise environment, is such a value that produces reliable rejection of impulses introducing minimal spectral distortions.

Above described robust DFT can be applied both to real and complex-valued signal $f(n)$ and noise $\nu(n)$ [4]. However, it is possible to pursue several alternatives in the case of complex-valued $x(n)$. Here we propose a novel modification of the robust DFT that is less sensitive to complex-valued impulse noise with mutually independent real and imagi-

nary parts. Potential applications of the proposed modification are in the fields where signal features are extracted from the spectrum of complex-valued signals, like for example: direction-of-arrival estimation of signals impinging on sensor arrays, coherent imaging systems [5] (including SAR), estimation of motion parameters in digital video-sequences processing, etc.

II. BREAKDOWN POINT ANALYSIS

Fundamental method for measurement of the transform robustness to impulse noise influence is so called breakdown point [6]. For a finite number of samples, the breakdown point can be defined as the smallest percentage of observations that must be replaced by arbitrary values in order to force an estimator to produce the values arbitrary far from the parameter values generated by non-noisy data.

The breakdown point in the L-filter based DFT for a real-valued noise is $bp(\alpha) = [N/2 - \alpha(N - 2)]/N$. Obviously, for the standard DFT, the breakdown point is $bp(0) = 1/N$, since a single sample can produce arbitrary estimate. For the median based estimate, the breakdown point is $bp(1/2) = 1/2$, i.e., at least half of samples should be corrupted by impulse noise in order to produce an arbitrary estimate. The breakdown point can be directly related to the number of impulses that can be rejected with the transform. Assume that we have a real valued noise $\nu(n)$, with probability of impulse appearance equal to p . Then, modulated signal sequence $x(n) \exp(-j\omega n)$ has the real part equal to $\text{Re}\{f(n) \exp(-j\omega n)\} + \nu(n) \cos(\omega n)$, while imaginary part is $\text{Im}\{f(n) \exp(-j\omega n)\} - \nu(n) \sin(\omega n)$. Probability of resulting impulse in both real and imaginary sequences is equal to p . The L-DFT with parameter α will reject impulses with percentage p for $bp(\alpha) > p$, i.e., α should be selected as:

$$\alpha < \frac{N(\frac{1}{2} - p)}{N - 2}. \quad (5)$$

However, parameter α that rejects impulses decreases in the case of complex-valued noise with mutually independent real and imaginary parts: $\nu(n) = \nu_1(n) + j\nu_2(n)$,

$E\{\nu_1(n)\nu_2(n)\} = 0$. Assume that the percentage of impulses in both real and imaginary parts is p . Then, resulting noise in the real part of the modulated signal sequence can be written as $\text{Re}\{\nu(n) \exp(-j\omega n)\} = \nu_1(n) \cos(\omega n) + \nu_2(n) \sin(\omega n)$. Under the considered assumptions, probability of impulse noise in resulting noise $\nu_1(n) \cos(\omega n) + \nu_2(n) \sin(\omega n)$ is approximately $2p - p^2$. The same situation holds in the case of imaginary part of modulated signal sequence. Then, elements from the sets $\mathbf{R}(\omega)$ and $\mathbf{I}(\omega)$ are corrupted by impulse noise with probability $2p - p^2$. In order to reject impulses, parameter α in the L-DFT should be selected as:

$$\alpha < \frac{N(\frac{1}{2} - 2p + p^2)}{N - 2}. \quad (6)$$

In order to illustrate values in (5) and (6), consider a typical example with calculation of the DFT with $N = 100$ samples and percentage of impulses of $p = 0.25$. Parameter α for real-valued noise can be selected as $\alpha < 0.255$ according to (5), while in the second case it is $\alpha < 0.064$, i.e., significantly smaller value of α should be selected. Recall that small α produces emphatic spectral distortion effects in the L-DFT [4]. To avoid this drawback, we propose a modification of the robust L-DFT for signals with independent real and imaginary disturbances in the next section.

III. PROPOSED MODIFICATION

In order to explain the proposed modification, the standard DFT is rewritten as:

$$\begin{aligned} X_S(\omega) &= \frac{1}{N} \sum_{n=0}^{N-1} [\text{Re}\{x(n)\} \\ &+ j\{\text{Im}\{x(n)\}\} \exp(-j\omega n) \\ &= R(\omega) + jI(\omega), \end{aligned} \quad (7)$$

where $R(\omega)$ and $I(\omega)$ are the standard DFTs of real and imaginary parts of signal $x(n)$, respectively:

$$R(\omega) = \frac{1}{N} \sum_{n=0}^{N-1} \text{Re}\{x(n)\} \exp(-j\omega n)$$

$$I(\omega) = \frac{1}{N} \sum_{n=0}^{N-1} \text{Im}\{x(n)\} \exp(-j\omega n). \quad (8)$$

For assumed noise model we can write that $\text{Re}\{x(n)\} = \text{Re}\{f(n)\} + \nu_1(n)$ and $\text{Im}\{x(n)\} = \text{Im}\{f(n)\} + \nu_2(n)$. One can easily draw a conclusion that both real and imaginary parts of modulated samples $\text{Re}\{x(n)\} \exp(-j\omega n)$ and $\text{Im}\{x(n)\} \exp(-j\omega n)$, used for evaluation of $R(\omega)$ and $I(\omega)$, are corrupted by impulses with probability p (not approximately $2p - p^2$ as in the case of $\text{Re}\{x(n) \exp(-j\omega n)\}$ and $\text{Im}\{x(n) \exp(-j\omega n)\}$). Then the L-statistics can be applied to both real and imaginary parts of modulated signal samples $\text{Re}\{x(n)\} \exp(-j\omega n)$ and $\text{Im}\{x(n)\} \exp(-j\omega n)$, with parameter α according to (5), i.e., with higher breakdown point than in the case of the original L-DFT.

This connection between the standard DFT of complex-valued signal and the standard DFTs of real and imaginary parts can be used for development of the modified L-DFT. The modified version of the L-DFT can be calculated as:

$$X'_L(\omega) = R_L(\omega) + jI_L(\omega), \quad (9)$$

where $R_L(\omega)$ and $I_L(\omega)$ are the L-filter forms of DFT calculated for $\text{Re}\{x(n)\} \exp(-j\omega n)$ and $\text{Im}\{x(n)\} \exp(-j\omega n)$:

$$R_L(\omega) = \sum_{m=0}^{N-1} a_m [\mathbf{r}'_m(\omega) + \mathbf{j}\mathbf{i}'_m(\omega)]$$

$$I_L(\omega) = \sum_{m=0}^{N-1} a_m [\mathbf{r}''_m(\omega) + \mathbf{j}\mathbf{i}''_m(\omega)], \quad (10)$$

where $\mathbf{r}'_m(\omega)$, $\mathbf{i}'_m(\omega)$, $\mathbf{r}''_m(\omega)$ and $\mathbf{i}''_m(\omega)$ are elements from the sets $\mathbf{r}'_m(\omega) \in \mathbf{R}'(\omega)$, $\mathbf{r}''_m(\omega) \in \mathbf{R}''(\omega)$, $\mathbf{i}'_m(\omega) \in \mathbf{I}'(\omega)$ and $\mathbf{i}''_m(\omega) \in \mathbf{I}''(\omega)$

$$\begin{aligned} \mathbf{R}'(\omega) &= \{\text{Re}\{\mathbf{x}(\mathbf{n})\} \cos(\omega \mathbf{n}) \text{ for } \mathbf{n} \in [\mathbf{0}, \mathbf{N}]\} \\ \mathbf{I}'(\omega) &= \{\text{Re}\{\mathbf{x}(\mathbf{n})\} \sin(\omega \mathbf{n}) \text{ for } \mathbf{n} \in [\mathbf{0}, \mathbf{N}]\} \\ \mathbf{R}''(\omega) &= \{\text{Im}\{\mathbf{x}(\mathbf{n})\} \cos(\omega \mathbf{n}) \text{ for } \mathbf{n} \in [\mathbf{0}, \mathbf{N}]\} \\ \mathbf{I}''(\omega) &= \{\text{Im}\{\mathbf{x}(\mathbf{n})\} \sin(\omega \mathbf{n}) \text{ for } \mathbf{n} \in [\mathbf{0}, \mathbf{N}]\}, \end{aligned} \quad (11)$$

sorted into the corresponding non-decreasing sequences. Probability that an element from

any of the sets $\mathbf{R}'(\omega)$, $\mathbf{R}''(\omega)$, $\mathbf{I}'(\omega)$ and $\mathbf{I}''(\omega)$ is corrupted by impulse noise is equal to p .

However, it seems that now two L-DFTs are evaluated ($R_L(\omega)$ and $I_L(\omega)$) for each frequency, causing increase of the calculation burden. Fortunately, the calculation complexity is practically not increased, since the modified L-DFT is evaluated only for $\omega \geq 0$, while it can be easily calculated for $\omega < 0$ as:

$$X'_L(\omega) = R_L^*(-\omega) + jI_L^*(-\omega). \quad (12)$$

Note that similar property holds for the standard DFT of real-valued signals since $X_S(\omega) = X_S^*(-\omega)$.

From this analysis it follows that the proposed modification produces higher breakdown point than the original robust L-DFT form, i.e., parameter α can be selected according to (5), in order to reject impulses with probability p in both real and imaginary parts. Also, calculation burden is not increased since two L-DFTs are evaluated only for $\omega \geq 0$.

IV. NUMERICAL ANALYSIS

Consider the test signal

$$f(t) = \exp(j \sin(8\pi(t/N)^2)), \quad (13)$$

where $t \in [0, 1]$ with $N = 300$ samples within the interval, embedded in an impulse noise with independent real and imaginary parts. The impulse noise is equal to either $-A$ or A with probability $a/2$, while value 0 is associated with probability $1 - a$ (in our experiments it is set that impulses have five times larger magnitude than the signal, i.e., $A = 5$). The mean-squared-error (MSE) is evaluated as a quality measure:

$$MSE_i(a, \alpha) = E\{|\Xi(\omega) - F(\omega)|^2\} \quad (14)$$

where $\Xi(\omega)$ is the transform of interest (L-filter DFT form or its modification) while $F(\omega)$ is the DFT of non-noisy signal. Index i in the MSE denotes the used L-DFT form: $i = 1$ is for L-DFT form (2), while $i = 2$ is for the proposed modification. Difference $MSE_1(a, \alpha) - MSE_2(a, \alpha)$ is depicted in Fig.1a. It can be seen that this function is positive-valued almost in entire domain. It means that the

proposed modification produces smaller MSE compared with the existing form. Also, it can be seen that enhancement is small for α close to 0.5 (both transforms approach the standard DFT) and for small a (small number of impulses in the signal). However, a significant improvement is obtained for high probability of impulses a and for small α (close to median form). These properties can be seen more clearly in Figs.1b-1e where the MSEs are depicted for:

- fixed $\alpha = 0.45$, close to the standard DFT where only small improvement is achieved by using the proposed modification;
- fixed $\alpha = 0.15$, close to median DFT where large improvement is obtained;
- small percentage of impulse $a = 10\%$ (the optimal value for the proposed transform is achieved for higher value of α than in the case of the original L-DFT);
- large number of impulses $a = 40\%$ with significant improvement achieved by the proposed transform and very accurate results for the wide region of parameter α values, $\alpha \in [0, 0.33]$.

Optimal value α_{opt} for a known percentage of impulses in the proposed L-DFT form is evaluated numerically as parameter α that minimizes the MSE for the considered a (Fig.2a). Numerically, we obtained linear ($\alpha_{opt}(a) = -0.56a + 0.48$) and quadratic ($\alpha_{opt}(a) = 0.26a^2 - 0.69a + 0.49$) interpolation for optimal value of α_{opt} as a function of probability a for the modified L-DFT. These expressions could be useful when percentage of impulses is known or accurately estimated. However, this rarely occurs in practice. Several various techniques are developed for adaptive estimation of the parameter α in the L-filters [7], [8]. Here we consider the modified Taguchi's approach described in [7], [9] for the adaptive α -trimmed mean filter as an example of adaptive procedure that can produce accurate results for the considered signal and noise model. Adaptive α parameter is evaluated as:

$$\alpha_{ad} = \frac{1}{2}E \left\{ \frac{|\hat{F}(\omega)|^2}{|X_S(\omega)|^2} \middle| \omega \in \Omega \right\}, \quad (15)$$

where $X_S(\omega)$ is the standard DFT of noisy signal, while $\hat{F}(\omega)$ is an estimate of the non-noisy

signal DFT and Ω represents considered frequency range. According to the Taguchi's recommendation, we adopt that $\hat{F}(\omega)$ is the corresponding robust L-DFT with $\alpha = 1/6$. For $|\hat{F}(\omega)| \approx |X_S(\omega)|$ it follows that $\alpha_{ad} \approx 1/2$, i.e., the L-DFT approaches the standard DFT since it can be assumed that this signal is not corrupted by impulse noise. For high noise influence $|\hat{F}(\omega)|^2 \ll |X_S(\omega)|^2$ it follows that $\alpha_{ad} \rightarrow 0$, i.e., the adaptive L-DFT is close to the median DFT form. In our experiments, the Taguchi's approach is applied to both robust L-DFT forms considered in the paper. The minimal MSEs achieved with the considered transforms and the MSE produced by the Taguchi's approach are given in Fig.2b as a function of a . It can be seen that the proposed modification outperforms the original robust DFT form. The Taguchi's approach applied to the original L-DFT form produces the MSE that is, on the average, 37% higher than the optimal transform (that could be evaluated for a known impulse noise probability). The Taguchi's approach applied to the proposed modification produces MSE just 4.5% higher than the optimal transform. In order to show this difference, small segment $\alpha \in [0.2, 0.3]$ is magnified in Fig.2b.

Finally, some numerical data for MSEs are summarized in Table I for various pairs (a, α) . The minimal MSE for the each considered noise parameter a is highlighted by boldface. Note that, MSE_2 is smaller than MSE_1 for $\alpha < 0.5$. Also, we would like to emphasize the fact that the proposed modification introduces considerably smaller MSE for small α than the original robust form in the case of non-noisy signal ($a = 0$), see second row in Table I. The MSE achieved with the Taguchi's approach is given in Table I, last column. It can be seen that for each considered impulse probability this approach applied to the modified L-DFT form produces very accurate results. Procedure for adaptive selection of the α -trimmed mean DFT parameter that can be used for arbitrary impulse noise environment will be topic of our further research.

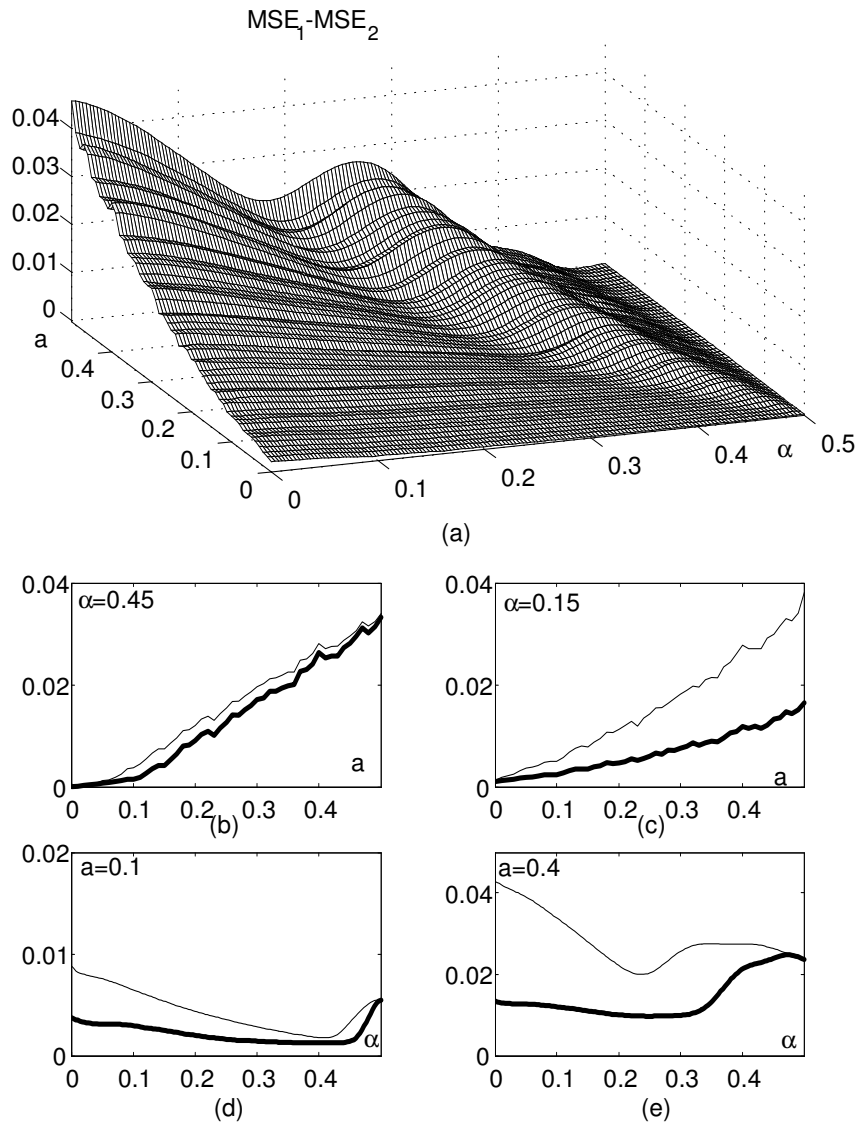


Fig. 1. Comparison of L-filter forms: (a) Difference $MSE_1(a, \alpha) - MSE_2(a, \alpha)$; (b) MSEs for fixed $\alpha = 0.45$; (c) MSEs for fixed $\alpha = 0.15$; (d) MSEs for fixed $a = 0.10$; (e) MSEs for fixed $a = 0.40$. Thin line - MSE_1 calculated for L-DFT; Thick line - MSE_2 calculated for proposed modification.

V. CONCLUSION

Modification of the robust L-filter DFT producing high breakdown point for complex-valued impulse noise environments with independent real and imaginary parts, without increase in calculation complexity, is proposed. Simple technique for estimation of the adaptive parameter in the L-filter DFT

form is introduced. Numerical examples confirm the presented theory. Application of the proposed modification to practical problems where the DFT is used to estimate parameters of complex-valued signals corrupted by impulse noise will be topic of our further research.

MSE ₁ /MSE ₂	$\alpha = 0.5$	0.4	0.3	0.2	0.1	0	Tag
$a = 0$	0/0	0.01/0.01	0.04/0.03	0.10/0.07	0.21/0.14	0.36/0.18	0.36/ 0
$a = 0.1$	0.58/0.58	0.19/ 0.13	0.28/0.15	0.43/0.20	0.62/0.29	0.84/0.35	0.37/0.16
$a = 0.2$	1.22/1.22	1.00/0.38	0.63/ 0.36	0.97/0.42	1.36/0.54	1.65/0.60	0.87/ 0.36
$a = 0.3$	1.84/1.84	1.89/1.19	1.24/ 0.64	1.54/0.69	2.23/0.85	2.81/0.92	1.67/ 0.64
$a = 0.4$	2.54/2.54	3.02/2.42	2.97/1.14	2.31/1.10	3.54/1.31	4.46/1.40	2.74/ 1.00
$a = 0.5$	3.04/3.04	4.28/3.25	4.28/2.02	3.78/1.60	4.91/1.83	6.49/1.98	4.09/ 1.53

TABLE I

MSE FOR TWO L-DFT FORMS. 1 - EXISTING; 2 - PROPOSED. THE BEST RESULTS ACHIEVED FOR CONSIDERED NOISE ENVIRONMENT ARE GIVEN IN BOLDFACE. TAG DEPICTS RESULTS OBTAINED WITH α_{ad} . VALUES IN THE CELLS ARE DIVIDED WITH 10^{-2} .

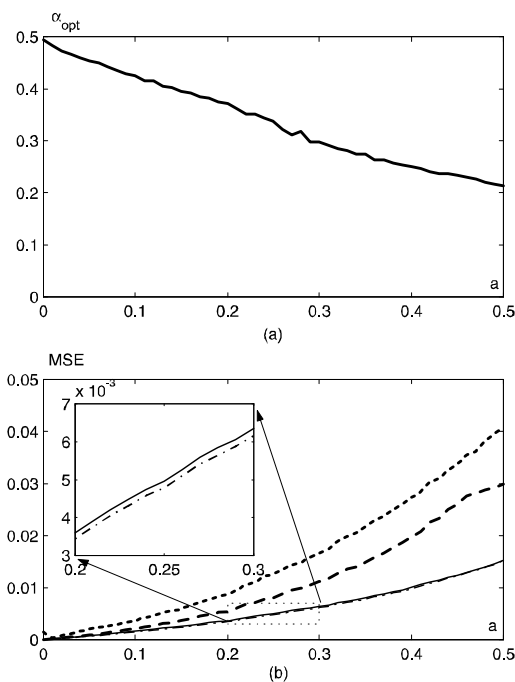


Fig. 2. Comparison of the optimal L-DFT and Taguchi's approach: (a) Optimal α as a function of a , $\alpha_{opt}(a)$, for the modified L-DFT form. (b) MSE of the L-DFT forms. Dash-dot line - proposed modification with $\alpha_{opt}(a)$; thick dashed line - original L-DFT form with $\alpha_{opt}(a)$; solid line - proposed modification with α_{ad} ; dotted line - original L-DFT form with α_{ad} . Small segment $a \in [0.2, 0.3]$ has been enlarged in order to show difference between MSE for modified L-DFT for optimal parameter α and the Taguchi's approach.

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