Realization of the Robust Filters in the Frequency Domain

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Abstract— **An efficient and simple procedure for filtering of signals in an impulse noise environment is proposed. It can be used for realization of all filter forms: lowpass, highpass, stopband, and bandpass. Accuracy of the proposed procedure is of the same order of magnitude as in the case of the weighted median/myriad filters admitting negative weights, recently proposed by Arce** ***et al.***

I. Introduction

Myriad and median filters are commonly used for the removal of impulse noise from lowpass data [1]-[3]. Recently, weighted forms of these filters, admitting negative weights, have been proposed [4]-[7]. They can produce all filter types (lowpass, highpass, stopband, and bandpass). Synthesis of these filters includes spectral optimization techniques [6] (or "training" procedures [4]). These procedures can be understood as synthesis of the filters robust to the impulse noise influence in the time-domain. The entire genesis of the weighted median/myriad filters development can be followed through [2]-[7].

Forms of the discrete Fourier transform (DFT), which can be used for spectral estimation in the case of signals embedded in the impulse noise, are introduced recently [8]. They are based on the same robust statistics concept as the median and myriad filters [9]. The robust M-DFT form, calculated by using the fixed-point search algorithm, is proposed in [8]. The marginal-median- filter-based form of the robust M-DFT is introduced in [10]. For the analysis of a mixture of the impulse and Gaussian noise, the L-filter- based DFT (L-DFT) forms are defined [11], [12]. In this paper, an implementation of robust filters in the frequency domain, based on the L-DFT, is proposed. Filtering is performed as a two-stage operation. In the first step, the L-DFT is calculated in order to produce an accurate estimate of the non-noisy signal DFT. The standard linear filtering procedure is performed in the second step. This quite simple procedure exhibits similar accuracy as the weighted median/myriad filters designed in the time-domain.

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The paper is organized as follows. A brief overview of the weighted median and myriad filter is given in Section II. Procedure for realization of the robust filters in the frequency-domain is described in Section III. Numerical study is presented in Section IV.

II. Weighted Median/Myriad Filters

Consider a signal $f(n)$ corrupted by a white noise $\nu(n)$, $x(n) = f(n) + \nu(n)$. The general form of the weighted median filter, admitting negative weights, follows as the solution of the minimization problem [4]-[7]:

$$\begin{aligned} s(n) &= \arg\min_{\theta} \sum\nolimits_{k=-N}^{N} [g(k)|x(n+k) - \theta| \\ &+ |h(k)||-x(n+k) - \theta|], \end{aligned} \tag{1}$$

while the general weighted myriad filter results from:

$$\begin{gathered} s(n) = \arg\min_{\theta} \sum\nolimits_{k=-N}^{N} \log[K^2 + g(k)\times \\ (x(n+k) - \theta))^2] \\ + \log[K^2 + |h(k)|(-x(n+k) - \theta))^2], \end{gathered} \tag{2}$$

where $g(k) \geq 0$ and $h(k) \leq 0$ for $k \in [-N, N]$, and real-valued K is the scaling or linearization factor. Standard weighted median/myriad smoothers follow from (1) and (2) for $h(k) = 0$, $k \in [-N, N]$. A proper choice of $g(k)$ and $h(k)$ in (1) and (2) can produce any spectral characteristics in the frequency domain, including the highpass or bandpass form [4], [6]. However, synthesis of these filters is not straightforward. It can be realized by using the generalized Mallow's algorithm [6], or iterative learning and training procedures [4]. These procedures represent digital filter synthesis in the time domain for the impulse noise environment. In order to simplify the calculation, the robust filter realization in the frequency domain is presented next.

III. Robust Filtering in the Frequency Domain

A. Robust DFT

A general form of the DFT can be defined as a solution of [8]

$$X_{F(e)}(\omega) = \arg\min_{\theta} \sum\nolimits_{n=0}^{N-1} F\left(x(n)e^{-j\omega n} - \theta\right). \tag{3}$$

where $\omega = 2\pi k/N$. The quadratic loss function $F(e) = |e|^2$ produces the standard DFT:

$$X_{|e|^2}(\omega) = \tfrac{1}{N} \sum\nolimits_{n=0}^{N-1} x(n)e^{-j\omega n}$$

$$= \text{mean}\{x(n)e^{-j\omega n} : n \in [0, N-1]\}. \tag{4}$$

The standard DFT is a poor estimate of the non-noisy signal DFT, even for a small amount of the impulse noise. The robust DFT is introduced in order to improve accuracy of the DFT estimate for the impulse noise environment. This transform follows from (3) for the loss function $F(e) = |e|$ [8]. It cannot be written as a closed form expression, and the iterative procedure should be used for its calculation. In order to avoid handling with the iterative procedure the "sub-optimal" marginal-median form of the robust DFT is introduced [10]:

$$X_{|e|}(\omega) =$$

$$\text{median}\{\text{Re}\{x(n)e^{-j\omega n}\} : n \in [0, N-1]\}$$

$$+j\text{median}\{\text{Im}\{x(n)e^{-j\omega n}\} : n \in [0, N-1]\}. \tag{5}$$

The robust DFT form (5) is a very accurate spectra estimate for a sum of sinusoids embedded in the impulse noise. However, it can produce distorted results for signals with varying spectral content.

B. L-filter Form of the DFT

The L-DFT forms are introduced as a tradeoff between robustness to the impulse noise influence and quality of the spectra estimate [11]. They can also be used in the case of signals embedded in a mixture of the Gaussian and impulse noise [1], [11]. The L-DFT is defined as [11], [12]

$$X_L(\omega) = \sum\nolimits_{n=0}^{N-1} a_n[\mathbf{r}_n(\omega) + j\mathbf{i}_n(\omega)], \tag{6}$$

where $\sum_{n=0}^{N-1} a_n = 1$, $\mathbf{r}_n(\omega)$ and $\mathbf{i}_n(\omega)$ represent values from the sets $\mathbf{R}(\omega) = \{\text{Re}\{x(n)e^{-j\omega n}\} : n \in [0, N-1]\}$ and $\mathbf{I}(\omega) = \{\text{Im}\{x(n)e^{-j\omega n}\} : n \in [0, N-1]\}$, $\mathbf{r}_n(\omega) \in \mathbf{R}(\omega)$, $\mathbf{i}_n(\omega) \in \mathbf{I}(\omega)$, $n \in [0, N-1]$ sorted into nondecreasing order: $\mathbf{r}_0(\omega) \leq \mathbf{r}_1(\omega) \leq ... \leq \mathbf{r}_{N-1}(\omega)$, $\mathbf{i}_0(\omega) \leq \mathbf{i}_1(\omega) \leq ... \leq \mathbf{i}_{N-1}(\omega)$. Here, we will use the α-trimmed mean form of the L-DFT, where coefficients a_n in (6) are given as:

$$a_n = \frac{1}{a(4-2N)+N}, \text{ for}$$

$$n \in [(N-2)a, (-N+2)a + N - 1], \tag{7}$$

and $a_n = 0$ elsewhere. This form of the DFT will be denoted by $X_a(\omega)$. It is equal to the standard DFT for $a = 0$, $X_0(\omega) = X_{|e|^2}(\omega)$, while for $a = 1/2$ it reduces to the marginal-median form (5), $X_{1/2}(\omega) = X_{|e|}(\omega)$. For $0 < a < 1/2$, this transform can give a tradeoff between spectra quality and impulse noise removal.

C. Filtering Procedure

A standard linear time-invariant filter can be represented in the frequency domain as: $S(\omega) = H(\omega)X(\omega)$, where $X(\omega)$ and $S(\omega)$ are the DFTs of the input and output signals respectively, while $H(\omega)$ is the frequency response. Since the L-DFT forms produce accurate estimates of the spectra for impulse noise environment, they can be used instead of the standard DFT. Then, the procedure for filtering is simple: a) Determine $X_a(\omega)$, [see (6) and (7)]; b) Multiply $H(\omega)$ by $X_a(\omega)$, where $H(\omega)$ is known in advance or determined through well-defined procedures in the case of linear filters; c) Calculate inverse discrete DFT

$$s(n) = N \sum_{k=0}^{N-1} H(2\pi k/N)\times X_a(2\pi k/N)\exp(j2\pi kn/N). \quad (8)$$

A problem that remains is the choice of the parameter a value in the α-trimmed mean filter form of the L-DFT (7), since it will be used as the DFT estimate. For a known signal model, this can be done through experiments and a training procedure. However, since the expected noise is of the impulse form and we want to preserve high quality of the spectra content, the α-trimmed form with $a \simeq 0.25$ can be used as an empirical trade-off between elimination of the impulse noise and spectra quality. Detailed analysis of this problem will be the topic of our further research. Note that the noise nature is more important factor for optimal parameter a choice in (7) than the particular signal shape. There are numerous methods for estimation of the pdf for the impulse kind of noise. For example, estimation of the parameters for the α-stable noises [13] can be found in [14] and references therein.

IV. Numerical Study

The same numerical example as in [4, Example 1] will be used. The chirp signal $f(n) = \sin(n\omega(n))$ is considered. Its frequency is varied according to the quadratic law $\omega(n) = 0.2\pi(n/(N-1))^2$, $n \in [0, N-1]$, and $N = 300$. The desired filter output (Fig.1a) is created by the FIR filter designed by using **fir1** function from MATLAB, with filter order 21 and passband cut-off frequencies $(\omega_1, \omega_2) = (0.16\pi, 0.18\pi)$. The output of the filter (8), with $a = 0.25$, is presented in Fig.1b. It can be seen that this signal is a good approximation of the desired signal from Fig.1a. Furthermore, signal $f(n)$ is embedded in the α-stable impulse noise [13] with $\alpha = 1.4$ and the dispersion factor $\gamma = 0.1$. One realization of the FIR filter output is shown in Fig.1c, while the output of the proposed filter with $a = 0.25$ is shown in Fig.1d. Note that **no training procedures for determination of the filter coefficients are used** here. The statistical analysis is performed for these two cases, as well. Non-noisy signal case is presented in Fig.2a. The MSE of the filter output is marked with a solid line as a function of a, while other lines represent the MSE obtained by using the *general myriad filter* (dashed line), and *special myriad filter* (dash-dot line) [4, Table II]. Noisy signal case is shown in Fig.2b. The dotted line represents results obtained by the FIR filter with coefficients determined from the least mean-square (LMS) algorithm [4]. For a noiseless signal the proposed procedure for $a = 0.25$ produces $MSE = 0.0059$. The general myriad filter behaves slightly worse ($MSE = 0.0060$) while the special myriad filter is the best ($MSE = 0.0033$). In total, 100 trials are considered for the case of a noisy signal. The proposed procedure for $a = 0.25$ gives $MSE = 0.0224$, which is slightly better than the special myriad filter with 0.0229 and slightly worse than the general myriad filter where $MSE = 0.0173$. It is significantly better than the FIR filter with LMS-trained coefficients, $MSE = 0.0544$. It can be seen that the MSE function is very smooth around its minimum ($MSE = 0.0198$ for $a = 0.12$). Thus, all values $a \in [0.05, 0.30]$ can be used for an accurate DFT estimate, producing accurate filter estimate. Note that as the amount of impulse noise increases, the higher values of the parameter a becomes "optimal". It means that the estimation of noise behavior is more important for the filter synthesis than the particular shape of signal. An interesting effect is that the MSE for the marginal median based robust DFT $a = 1/2$ is lower in the case of the impulse noise than in non-noisy case (see Figs.2b, and 2a, respectively). As expected, for the standard DFT, $a = 0$, $MSE = 0$ for nonnoisy signal (see Fig.2a) while the MSE assumes a very high value for the signal with impulse noise (see Fig.2b).

Finally, we considered noisy signal with varying α parameter and fixed $\gamma = 0.1$. Fig.2c shows the MSE, as a function of (a, α). The minimal observed MSE for this noise case was 0.0186. Three isolines are presented $[0.022, 0.03, 0.05]$. It can be seen that the standard DFT becomes useless very fast when the impulse noise amount increases, i.e., when α decreases. There is a large portion of the (a, α) plane centered around $a = 0.2$ where

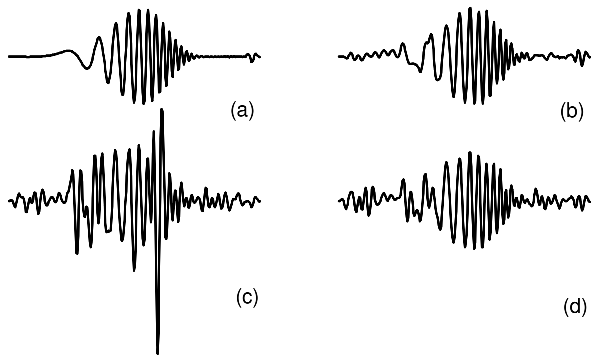

Fig. 1. Bandpass filtering: (a) Desired signal; (b) Robust filter for $a = 0.25$ of noiseless signal; (c) FIR filtering of the noisy signal; (d) Robust filter for $a = 0.25$ of noisy signal.

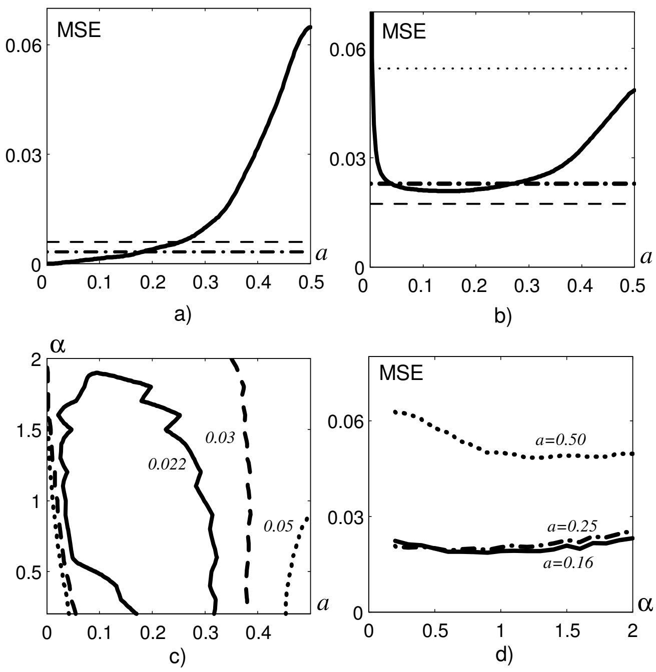

Fig. 2. MSE of the filter output: (a) Noiseless signal; (b) Noisy signal $\alpha = 1.4$, $\gamma = 0.1$. Solid line represents the proposed filter. The dotted line represents the FIR filter with the LMS training. The dashed line represents the general myriad filter. The dash-dot line represents the special myriad filter; (c) Contour plots of the MSE for $[0.022, 0.03, 0.05]$ as a function of (α, a); (d) MSE as a function of α for $a \in [0.16, 0.25, 0.5]$.

the MSE is almost constant. This performance can be slightly different for other signal types but our numerical experiments confirmed that, roughly speaking, $a \in [0.10, 0.30]$ produces satisfactory accuracy in all cases. The MSE error, as a function of the impulse noise amount, is shown in Fig.2d for the robust DFT $a = 0.5$, and the L-DFTs for $a = 0.25$ and $a = 0.16$. Note that $a = 0.16$ produces the smallest 'mean MSE' for the presented signal calculated over α.

V. Conclusion

Frequency domain filtering of signals corrupted by the impulse noise is presented. The basic filtering tool is the L-DFT, which can remove a significant part of the impulse noise. The second step in filtering is a well known standard linear filtering realization. Results obtained with this simple procedure are slightly worse than those produced with the weighted median/myriad filters, but at the same time the algorithm complexity is significantly reduced. Accurate results are obtained for relatively wide range of the parameter a values. Algorithms for the determination of optimal parameter in the L-DFT calculation are the topic of our further research.

References

[1] I. Pitas and A.N. Venetsanopoulos: "Order statistics in digital image processing," *Proc. IEEE*, Vol.80, No.12, Dec. 1992, pp.1893-1992.

[2] S. Kalluri and G.R. Arce: "Adaptive weighted myriad filter algorithms for robust signal processing in α-stable noise environment," *IEEE Trans. Sig. Proc.*, Vol.46, No.2, Feb. 1998, pp.322-334.

[3] S. Kalluri and G.R. Arce: "Fast algorithms for weighted myriad computation by fixed-point search," *IEEE Trans. Sig. Proc.*, Vol.48, No.1, Jan. 2000, pp.159-171.

[4] S. Kalluri and G.R. Arce: "Robust frequency-selective filtering using weighted myriad filters admitting real-valued weights," *IEEE Trans. Sig. Proc.*, Vol.49, No.11, Nov. 2001, pp.2721-2733.

[5] G.R. Arce: "A general weighted median filter structure admitting negative weights," *IEEE Trans. Sig. Proc.*, Vol.46, No.12, Dec. 1998, pp.3195-3205.

[6] I. Shmulevich and G.R. Arce: "Spectral design of weighted median filters admitting negative weights," *IEEE Sig. Proc. Let.*, Vol.8, No.12, Dec. 2001, pp.313-316.

[7] G.R. Arce and J.L. Parades: "Recursive weighted median filters admitting negative weights and their optimization," *IEEE Trans. Sig. Proc.*, Vol.48, No.3, Mar. 2000, pp.768-779.

[8] V. Katkovnik: "Robust M-periodogram," *IEEE Trans. Sig. Proc.*, Vol.46, No.11, Nov. 1998, pp.3104-3109.

[9] P.J. Huber: *Robust statistics*, John Wiley&Sons, 1981.

[10] I. Djurović, V. Katkovnik and LJ. Stanković: "Median filter based realizations of the robust time-frequency distributions," *Sig. Proc.*, Vol.81, No.7, July 2001, pp.1771-1776.

[11] I. Djurović, LJ. Stanković and J.F. Böhme: "Estimates of the Wigner distribution in Gaussian noise environment," *AEU Int. Jour. El. Comm.*, Vol.56, No.5, pp.377-340, 2002.

[12] I. Djurović, LJ. Stanković and J.F. Böhme: "Robust L-estimation based forms of signal transforms and time-frequency representations," *IEEE Trans. Sig. Proc.*, Vol. 51, No. 7, July 2003, pp. 1753-1761.

[13] C.L. Nikias and M. Shao: *Signal processing with alpha stable distributions and applications*, New York: Willey 1995.

[14] S. Maymon, J. Friedmann and H. Messer: "A new method for estimating parameters of a skewed aplha-stable distribution," in *Proc. of IEEE ICASSP'2000*, Vol.6, June 2000, pp.3822-3825.