

Single-Iteration Algorithm for Compressive Sensing Reconstruction

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Abstract — In the light of popular compressive sensing concept, this paper proposes a single-iteration reconstruction algorithm for recovering sparse signals from its incomplete set of observations. Compressive sensing assumes that a signal which is sparse in certain transform domain can be randomly sampled in another (dense) domain, taking lower number of samples than required by the sampling theorem. Then, using the optimization algorithms, the entire signal information can be recovered. In our case, instead of using ℓ_1 -based methods or approximate greedy solutions, we propose a simple algorithm based on the analysis of noisy-effects that appear in the sparsity domain as a consequence of missing samples. The theory is proven on the examples.

Keywords — Compressive sensing, DFT, sparsity, reconstruction algorithms

I. INTRODUCTION

Nowadays, one of the most challenging topics and a key issue in various applications such as data compression, source separation, noise reduction, and more recently compressed sensing is finding a sparse linear decomposition of a given signal [1]-[3]. Many of the proposed algorithms provides good sparse approximations in polynomial time, but the general problem of finding the best m -term approximation is non-polynomial (NP-Complete). When searching for the best sparse approximation, we can define two general and most popular approaches [4]-[7]. The first one is l_1 minimization approach which is solved using convex optimization algorithms [4],[5]. On the other hand, there are much faster so called greedy algorithms, which are iterative procedure that decrease the approximation error by relaxing the sparsity constraint. The most commonly used among them is the Orthogonal Matching Pursuit (OMP) [6],[7], together with a number of its variations such as Gradient Pursuit (GP), CoSaMP, etc. Although they are faster and simpler than convex optimization algorithms, the greedy ones usually assume that the number of signal components that should be reconstructed is a priori known. Otherwise, the iterations stopping criterion should be based on mean

square error [8], which should be specified in advanced and estimated empirically in each application. Instead of this, we propose a single-iteration solution which is based on the analysis of effects caused by missing samples in the observation domain. We start from the assumption that the observed signals are sparse in the Fourier transform domain. The side effects caused by having an incomplete set of time observations instead of full data set are modeled as spectral noise characterized by a mean value and variance. As long as we are able to keep the noise effects below the signal components level in the Fourier domain, we can provide the single-iteration reconstruction algorithm. The proposed solution also defined an optimal number of available samples called measurements which is required for achieving the reconstruction with low probability of error. Here, it is important to emphasize that the proposed concept is not only applicable to the signal and its Fourier transform, but can be extended to the time-frequency representations and distributions that have been crucial in various real applications including radars [9], biomedicine [10], multimedia [11],[12], hardware and instrumentation [13]-[15], etc. Namely, most of the existing time-frequency representations are obtained as the Fourier transform of windowed linear, bilinear, polynomial, or complex-lag auto-correlation function [16]-[21]. In this case, the proposed concept should be applied to each windowed auto-correlation function segment.

The paper is organized as follows. The theoretical background about the compressive sensing is given in Section II. The single iteration reconstruction algorithm is proposed in Section III. The experimental results are presented in Section IV, while the Concluding remarks are given in Section V.

II. COMPRESSIVE SENSING

When dealing with signal that, according to the sampling theorem, needs a high sampling rate, the acquisition of samples requires a high number of sensors, as well as large data storage and transmission capacities. In these circumstances it would be very feasible to explore the possibility of sampling at far lower rates and afterwards, or upon the occasion, to reconstruct the rest of the signal for the purpose of analysis, representation, etc. In order to achieve this goal, we usually need to sample randomly and to identify the domain of signal sparsity to allow signal reconstruction. The mathematical foundation of CS lies in the fact that it is possible to reconstruct a sparse signal

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from an underdetermined linear system of equations and that this can be done in a computationally efficient manner via convex programming [5]. Consider a signal $s \in \mathbb{C}^N$ that can be represented in certain basis $\Psi = \{\Psi_k\}_{k=1}^N$, using the weighting coefficients x_k :

$$s = \sum_{k=1}^N x_k \Psi_k .$$

When observing in the vector form we can write:

$\mathbf{s} = \Psi \mathbf{x}$, where $\Psi = [\Psi_1, \Psi_2, \dots, \Psi_N]$ is a full rank $N \times N$ matrix. One such example can be a finite length, discrete signal that can be represented using discrete sinusoids in a limited bandwidth. The matrix Ψ would then be the discrete Fourier transform (DFT) matrix. In compressive sensing one is particularly interested in any basis that allows a sparse representation of \mathbf{s} , i.e., a basis Ψ such that most of the values within \mathbf{x} are zero. In compressive sensing we assume that \mathbf{s} is not available. Instead of the whole signal \mathbf{s} we are actually dealing with a small set of M randomly sensed measurements, where $M \ll N$. A set of random measurements are selected from signal \mathbf{s} , by using random measurement matrix Φ [1]-[3]:

$$\mathbf{y} = \Phi \mathbf{s} . \quad (1)$$

Accordingly, we may write:

$$\mathbf{y} = \Phi \Psi \mathbf{x} = \Theta \mathbf{x} . \quad (2)$$

In order to obtain the reconstructed signal version, we have to solve the underdetermined system of M linear equations with N unknowns. It is obvious that this system may have infinitely many solutions, but the idea is to search for the sparsest one. For this purpose, various optimization algorithms based on ℓ_0 -norm minimization have been used. However, due to the problem complexity, the ℓ_0 -norm minimization have been replaced in practical applications with the ℓ_1 -norm minimization, leading to a near-optimal solutions. The ℓ_1 -norm minimization problem in compressive sensing can be defined as follows [4]:

$$\min \|\tilde{\mathbf{x}}\|_{\ell_1} \quad s.t. \quad \mathbf{y} = \Theta \tilde{\mathbf{x}} . \quad (3)$$

The above minimization can be solved by using convex optimization algorithms such as Basis Pursuit algorithm, with some of the commonly used solvers such as simplex and interior point methods (e.g., primal-dual interior point method). However, the complexity of such realization are still quite high. Therefore, for a real-time processing the so called Greedy algorithms have been introduced, such as MP or OMP, which represents an iteratively approximate solution that includes the best fitting component in each iteration [6]-[8]. They are much less complex than convex optimizers but are not guaranteed to converge to the optimal solution.

III. SINGLE-ITERATION RECONSTRUCTION ALGORITHM (SIRA)

In this Section we propose a single-step (non-iterative) that is based on signal sparsity in the DFT domain. For that

purpose we need to characterize the side-effects that appear as a consequence of having incomplete set of signal samples. Namely, the missing samples in the time domain (observation domain) will produce a certain kind of noise in the spectral domain (Fourier transform, i.e., domain of sparsity) which deteriorates signal representation. Larger number of missing samples produces larger noise variance, ruining the signal's sparsity. Consequently, signal components detection becomes more difficult. Hence, to provide an automatic components detection and reconstruction when dealing with compressive sensed signal, we need to start from the expression which relates number of missing samples to the spectral noise variance. Let assume that signal $\mathbf{s} = s(n)$, $n=1, \dots, N$ consists K sinusoidal frequency components defined by the amplitudes A_i and frequencies k_i , $i=1, \dots, K$. The DFT of this signal is defined as:

$$S(k) = \sum_{n=0}^{N-1} \sum_{i=0}^K A_i e^{-2\pi j(k-k_i)n/N} , \quad (4)$$

From the previous relation we can observe the set of samples \mathbf{h} defined as:

$$h(n) = \left\{ \sum_{i=1}^K A_i e^{-2\pi j(k-k_i)n/N} , n = 0, \dots, N-1 \right\} ,$$

where $\sum_{n=0}^{N-1} h(n) = 0$ holds.

Furthermore, consider a set \mathbf{y} of M available samples from \mathbf{h} corresponding to the CS signal. The DFT over the available set of samples can be written as follows:

$$F(k) = \sum_{n=1}^M y(n) = \sum_{n=1}^M h(n) - v(n) \quad (5)$$

where at the positions of missing samples the noise can be modeled as $v(n) = h(n)$. The mean value of F can be calculated as:

$$E\{F\} = \sum_{i=1}^K M A_i \delta(k - k_i) , \quad (6)$$

whereas the variance of DFT values at the non-signal and signal positions can be calculated according to [22]:

$$\sigma^2 = \text{var}\{F_{k \neq k_i}\} = \frac{M(N-M)}{(N-1)} \sum_{k=1}^K A_i . \quad (7)$$

The DFT values at the non-signal positions (noise-alone positions) are Rayleigh distributed:

$$f(z) = \frac{z}{\sigma^2} e^{-\frac{z^2}{2\sigma^2}} . \quad (8)$$

Since the DFT values of the i -th signal component is equal to MA_i , then using the Rayleigh distribution, we can now define the probability that all noise-alone DFT values are below a certain DFT component MA_i :

$$P_e = 1 - \left(1 - e^{-\frac{|X_i|^2}{\sigma^2}} \right)^{N-K} \approx 1 - \left(1 - e^{-\frac{MA_i^2}{\sigma^2}} \right)^N \quad (9)$$

TABLE 1: THE PROPOSED ALGORITHM FOR SIGNAL RECONSTRUCTION

Single-iteration reconstruction algorithm SIRA

1. Set desired P_e	Set high value for the probability P, e.g. $P=10^{-2}$
2. $M_{opt} \geq \arg \min \{P_e\}$	Calculate optimal number of available samples using $A_i=A_{min}$ (the lowest amplitude of components)
3. $\mathbf{X}=DFT\{\mathbf{y}\}$	Calculate the initial DFT vector \mathbf{X} that corresponds to the set of M_{opt} available measurements
4. $\mathbf{k}=\arg\left\{ \mathbf{X} >\sqrt{-\sigma^2 \log(1-\sqrt[N]{P_e})}\right\}$	Find vector \mathbf{k} of positions of DFT components higher than $\sqrt{-\sigma^2 \log(1-\sqrt[N]{P_e})}$
5. $\mathbf{X}=(\Theta^* \Theta)^{-1} \Theta^* \mathbf{y}$	Exact DFT values at positions \mathbf{k} are obtained. CS matrix Θ is obtained from the DFT matrix using rows that correspond to the frequencies \mathbf{k} and columns corresponding to M available measurements. The system is solved in the least square sense.

The previous expression can be used as an approximate form of error probability P_e (probability of wrong detection of the i -th signal component), which will be a basis of signal reconstruction algorithm summarized below. Here, we may assume that K is negligible since $K \ll N$ holds.

IV. EXAMPLES

Example 1: Let us observe a multicomponent signal given in the form:

$$s(n) = \sum_{i=1}^{14} A_i \exp(j2\pi n f_i / N)$$

where the components amplitudes are given as follows: $A_1=3.5, A_2=3, A_3=2, A_4=1.75, A_5=4, A_6=3, A_7=2.5, A_8=3.2, A_9=3.75, A_{10}=3.4, A_{11}=2, A_{12}=2.3, A_{13}=3, A_{14}=3.3$, while the frequencies of the components are:

$f_1=32, f_2=38, f_3=50, f_4=128, f_5=136, f_6=140, f_7=148, f_8=256, f_9=272, f_{10}=280, f_{11}=400, f_{12}=415, f_{13}=426, f_{14}=435$. The total number of samples is $N=512$, while the number of available samples is $M_{opt}=230$. The probability is set to $P_e=0.01$. Note that M_{opt} is calculated for the lowest signal amplitude $A_5=1.75$, according to: $M_{opt} \geq \arg \min \{P_e\}$ for a fixed $P_e=0.01=10^{-2}$ as illustrated in Fig. 1. The single-iteration solution provided by Algorithm 1 is applied. The initial DFT vector \mathbf{X} is shown in Fig. 2, while correctly reconstructed DFT signal components are shown in Fig. 3.

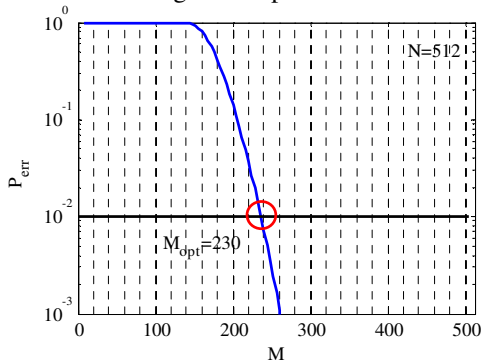


Fig. 1. Optimal number of available samples which assures $P_e=10^{-2}$

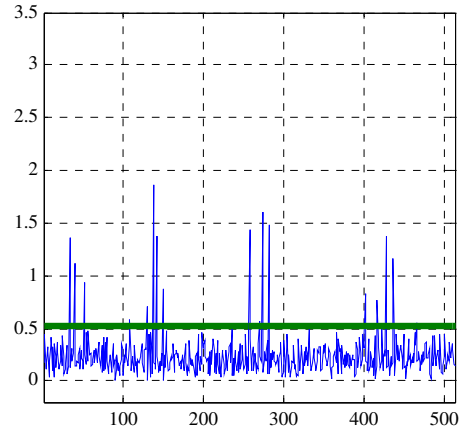


Fig. 2. Initial DFT: finding positions within DFT vector \mathbf{X} with values higher than $\sqrt{-\sigma^2 \log(1-\sqrt[N]{P_e})}$ (marked by horizontal line)

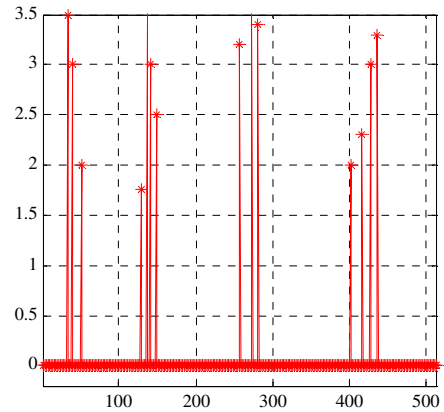


Fig. 3. Reconstructed DFT components

Example 2: In this example we observe the application of the proposed SIRA reconstruction algorithm to the time-frequency representation. For the sake of simplicity we observe the short-time Fourier transform case (STFT) applied to the signal in the form:

$$s = 3e^{32j\pi t} + 2.5e^{60j\pi t} + 2e^{128j\pi t}$$

The window width used in the STFT calculation is 128 samples. Assuming that only 40% of samples are available within each window, we have calculated the STFT shown in Fig. 4. Note that due to the missing samples, the STFT

has visible drawbacks reflecting as a noise in the time-frequency plane. Therefore, we need to apply the proposed reconstruction algorithm separately to each windowed signal part in order to reconstruct the entire signal portion (as it is done in the case of DFT in previous example). The results of reconstruction are shown in Fig. 5. Note that the sinusoids are perfectly reconstructed.

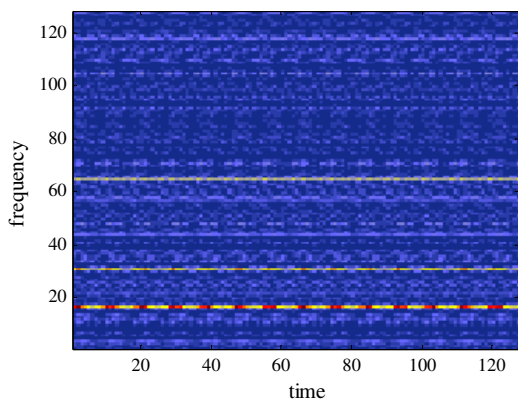


Fig 4. The time-frequency representation obtained using the STFT based on the available set of samples

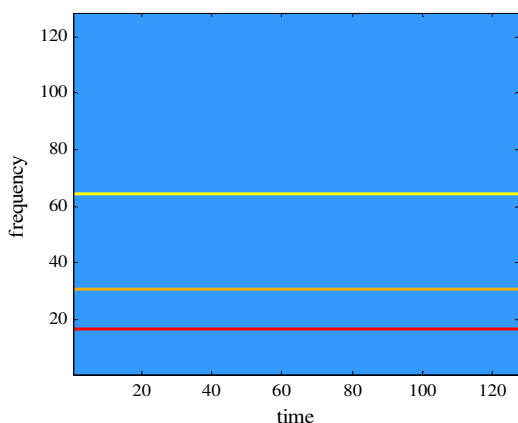


Fig 5. The Reconstructed STFT version using the proposed SIRA algorithm

V. CONCLUSION

The single-iteration algorithm for compressive sensing reconstruction is proposed. Unlike most of the previously designed algorithms, the proposed solution is able to detect and reconstruct all signal components at once. For that purpose, we need to calculate the optimal number of available measurements that will assure total signal reconstruction with desired low probability of error. The proposed method focuses to signals that are sparse in the DFT domain, but it has been shown that the proposed concept can be used even for time-frequency representations, where we might assume that the signal is sparse within the windowed segments.

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