

# From the STFT to the Wigner Distribution

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Analysis, processing and parameters estimation of signals whose spectral content changes in time are of crucial interest in many applications, including radar, acoustics, biomedicine, communications, multimedia, seismic, and car industry [1]- [11]. Various signal representations have been introduced to deal with this kind of signals within the area known as **time-frequency (TF) signal analysis**. The oldest analysis tool in this area is the short-time Fourier transform (STFT), as a direct extension of the classical Fourier analysis. The other key tool is the Wigner distribution, introduced in signal analysis from quantum mechanics. The aim of this note is to present and relate these two of the most important tools in the TF signal analysis, the STFT and the Wigner distribution (introduced by two Nobel prize winners, D. Gabor and E. Wigner, respectively). This relation is a basis for the S-method, an efficient and simple TF signal analysis tool providing a gradual transition between these two representations.

## I. RELEVANCE

Time-frequency analysis is of great importance to researchers, students and engineers dealing in their work or research with processing of signals with time-varying spectra. Two main approaches are used in TF signal analysis. One of them is based on the STFT and its variations and parameter optimizations. The other is based on the Wigner distribution, including its cross-terms reduced forms, defined through a general Cohen class of distributions. Since most of the classical time-frequency analysis tools are based on the STFT calculation, an approach that can simply and efficiently upgrade the existing STFT based systems toward higher concentrated Wigner distribution forms is of great practical and theoretical significance.

## II. PREREQUISITES

This article assumes a basic knowledge of the linear algebra and the Fourier

transforms, including discrete Fourier transform (DFT).

## III. PROBLEM STATEMENT

Present a direct relation between two of the most important tools in the TF signal analysis, the STFT and the Wigner distribution. Using this relation, implement a gradual transition from the STFT toward the Wigner distribution. Produce a time-frequency representation, combining good properties of the cross-terms free STFT and the highly concentrated Wigner distribution.

## IV. SHORT-TIME FOURIER TRANSFORM

Basic idea behind the STFT, as the initial and the simplest TF representation, is to apply the Fourier transform (FT) to a localized (truncated) signal  $x(t)$ , obtained by using a sliding window function  $w(t)$ . It is defined by

$$S(t, \Omega) = \int_{-\infty}^{\infty} x(t + \tau)w(\tau)e^{-j\Omega\tau}d\tau. \quad (1)$$

It is clear that the STFT satisfies properties inherited from the FT.

In the discrete time-frequency domain the STFT, at an instant  $n$  and frequency  $k$ , reads

$$S_N(n, k) = \sum_{m=-\frac{N}{2}}^{\frac{N}{2}-1} x(n+m)w(m)e^{-j\frac{2\pi}{N}mk}.$$

The STFT  $S_N(n, k)$  is calculated using signal samples within the window  $[n - N/2, n + N/2 - 1]$  for  $-N/2 \leq k \leq N/2 - 1$ , corresponding to an even number of  $N$  discrete frequencies from  $-\pi$  to  $\pi$ . For an odd  $N$  the summation limits are  $\pm(N-1)/2$ . A wide window includes signal samples over a wide time interval, losing the possibility to detect fast changes in time, but achieving high frequency resolution. A narrow window in the STFT will track time changes, but with a small resolution in frequency. Two extreme cases are  $N=1$  when  $S_1(n, k) = x(n)$  and  $N=M$  with  $S_M(n, k) = X(k)$ , where  $M$  is the number of all available signal samples and

$X(k) = \text{DFT}\{x(n)\}$  is a discrete FT of the signal.

Constant or, in general, varying window widths  $N_i$  could be used for different time instants  $n_i$ . Assuming a rectangular window we can write,

$$S_{N_i}(n_i, k) = \sum_{m=-\frac{N_i}{2}}^{\frac{N_i}{2}-1} x(n_i+m)e^{-j\frac{2\pi}{N_i}mk}$$

$$\mathbf{S}_{N_i}(n_i) = \mathbf{W}_{N_i}\mathbf{x}_{N_i}(n_i), \quad (2)$$

where  $\mathbf{S}_{N_i}(n_i)$  and  $\mathbf{x}_{N_i}(n_i)$  are column vectors with elements  $S_{N_i}(n_i, k)$ ,  $k = -N_i/2, \dots, N_i/2 - 1$  and  $x(n_i + m)$ ,  $m = -N_i/2, \dots, N_i/2 - 1$ , respectively. Matrix  $\mathbf{W}_{N_i}$  is an  $N_i \times N_i$  DFT matrix with elements  $\exp(-j2\pi mk/N_i)$ , where  $m$  is the column index and  $k$  is the row index of the matrix. The STFT value  $S_{N_i}(n_i, k)$  is presented as a block in the TF plane of the width  $N_i$  in the time direction, covering all time instants  $[n_i - N_i/2, n_i + N_i/2 - 1]$  used in its calculation. The frequency axis can be labeled with the DFT indices  $p = -M/2, \dots, M/2 - 1$  corresponding to the DFT frequencies  $2\pi p/M$  (blue dot positions in Fig.1a, b)). With respect to this axis labeling, the block  $S_{N_i}(n_i, k)$  will be positioned at the frequency  $2\pi k/N_i = 2\pi(kM/N_i)/M$ , i.e., at  $p = kM/N_i$ . The block width in frequency is  $M/N_i$  DFT samples. Therefore the block area in time and DFT frequency is always equal to the number of all available signal samples  $M$ , Fig.1.

When  $N_i$  changes with  $n_i$  we have the case of a **time-varying window**. In a nonoverlapping STFT, covering all signal samples  $\mathbf{x} = [x(0), x(1), \dots, x(M-1)]^T$  with  $\mathbf{S}_{N_i}(n_i)$ , the STFT should be calculated at  $n_0 = N_0/2$ ,  $n_1 = N_0 + N_1/2$ ,  $n_2 = N_0 + N_1 + N_2/2, \dots, n_K = M - N_K/2$ . A matrix form for all STFT values is

$$\mathbf{S} = \begin{bmatrix} \mathbf{W}_{N_0} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{W}_{N_1} & \dots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{W}_{N_K} \end{bmatrix} \mathbf{x}$$

$$\mathbf{S} = \tilde{\mathbf{W}}\mathbf{x} = \tilde{\mathbf{W}}\mathbf{W}_M^{-1}\mathbf{X}, \quad (3)$$

where  $\mathbf{S}$  is a column vector containing all STFT vectors  $\mathbf{S}_{N_i}(n_i)$ ,  $i = 0, 1, \dots$ ,

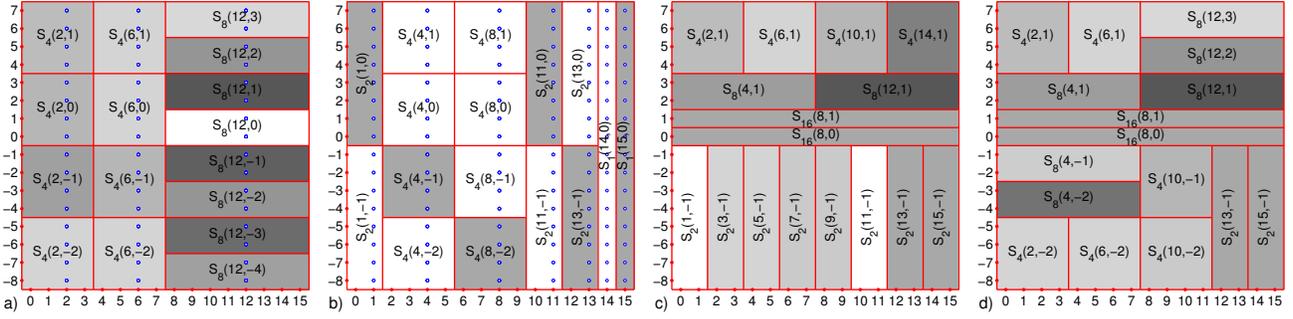


Fig. 1. The STFTs with a: (a), (b) time-varying window, (c) frequency-varying window, and (d) time-frequency-varying window. Time index is presented on the horizontal axis, while the DFT frequency index is shown on the vertical axis.

$K$ ,  $\mathbf{X} = \mathbf{W}_M \mathbf{x}$  is a DFT of the whole signal  $x(n)$ , while  $\tilde{\mathbf{W}}$  is a block matrix ( $M \times M$ ) formed from the smaller DFT matrices  $\mathbf{W}_{N_0}, \mathbf{W}_{N_1}, \dots, \mathbf{W}_{N_K}$ , as in (2). Since the time-varying nonoverlapping STFT corresponds to a decimation-in-time DFT scheme, its calculation is more efficient than the DFT calculation of the whole signal. Illustration of time-varying window STFTs is shown in Fig.1(a), (b). For a signal with  $M$  samples, there is a large number  $F(M) = 2^{M-1}$  of possible nonoverlapping STFTs with a time-varying window  $N_i \in \{1, 2, 3, \dots, M\}$ , [12]. A simple way to compare various STFTs from the concentration point of view is described in *Example 1*.

In general, for a nonrectangular window, (2) is slightly modified as  $\mathbf{S}_{N_i}(n_i) = \mathbf{W}_{N_i} \mathbf{H}_{N_i} \mathbf{x}_{N_i}(n_i)$ , where  $\mathbf{H}_{N_i}$  is a diagonal  $N_i \times N_i$  matrix with the window values on the diagonal,  $H_i(m, m) = w_i(m)$ ,  $m = -N_i/2, \dots, N_i/2 - 1$ . In a full matrix notation, for all nonoverlapping instants  $n_i$ , we get

$$\mathbf{S} = \tilde{\mathbf{W}} \tilde{\mathbf{H}} \mathbf{x}, \quad (4)$$

where  $\tilde{\mathbf{W}}$  and  $\tilde{\mathbf{H}}$  are  $M \times M$  matrices formed from smaller  $N_i \times N_i$  matrices  $\mathbf{W}_{N_i}$  and  $\mathbf{H}_{N_i}$ , respectively, as in (3). Another way of composing STFTs calculated with a rectangular window into a STFT with, for example, the Hann(ing), Hamming, or Blackman window, is presented in *Example 2*.

The STFT may use **frequency-varying window** as well. For a given DFT frequency  $p_i$  the window width in time is constant, Fig.1(c). Combining time-varying and frequency-varying windows we get **hybrid TF-varying windows** with  $S_{N_i(k_i)}(n_i, k_i)$ , Fig.1(d). They are efficiently used for adaptive

estimation and filtering. For a graphical representation of the STFT with varying windows, the corresponding STFT value should be assigned to each instant  $n = 0, 1, \dots, M - 1$  and each DFT frequency  $p = -M/2, -M/2 + 1, \dots, M/2 - 1$  within a block. In the case of a hybrid TF-varying window the matrix form is not obtained from blocks, as in the time-varying case, but writing expression for each STFT value. For example, for the STFT calculated as in Fig.1(d), for each STFT:  $S_4(2, -2)$ ,  $S_4(6, -2)$ ,  $S_4(10, -2)$ ,  $S_2(13, -1)$ , ...,  $S_8(12, 3)$ , an expression based on (2) should be written. Then the resulting matrix  $\mathbf{S} = \tilde{\mathbf{W}} \mathbf{x}$  can be formed.

Nonoverlapping cases are important and easy for analysis. They also keep the number of the STFT coefficients equal to the number of the signal samples. However there several reasons for introducing **overlapped STFT** representations. Rectangular windows have poor localization in the frequency domain. Study of the well-localized window forms in the TF domain has been an important topic, since the STFT concept was introduced. In the case of nonrectangular windows some of the signal samples are weighted in such a way that their contribution to the final representation is small. Then we want to use at least one more STFT with a window centered at these samples. Also, in the parameters estimation and detection the task is to achieve the best possible estimation or detection for each time instant instead of using interpolations for the skipped instants. Commonly, the overlapped STFTs are calculated using, for example, rectangular, Hann(ing), Hamming, Bartlett, Kaiser, or Blackman window of a constant window width  $N$

with steps  $N/2, N/4, N/8, \dots$  in time. Computational cost is increased in the overlapped STFTs. Analysis of overlapping cases may be considered as a superposition of the nonoverlapping cases (*Example 2*).

Dimensions of the STFT blocks (resolutions) are determined by the window width. The best STFT for a signal would be the one whose window form fits the best to the signal's TF content. Consider, for example, an important and simple signal such as a linear frequency modulated (LFM) chirp. For simplicity of analysis assume that its IF coincides with the TF plane diagonal. It is obvious that, due to symmetry, both time and frequency resolution are equally important. Therefore, the best STFT would be the one calculated by using a constant window whose block has an equal number of samples in time and DFT frequency,  $N_i = M/N_i$  [4], [12]. For this LFM signal, for example, with  $M = 256$  samples in total, the best choice would be the STFT with  $N_i = \sqrt{M} = 16$ . With such a window both resolutions will be the same and equal 16. These resolutions could be unacceptably low for many applications. It means that the STFT, including all of its possible time and/or frequency-varying window forms, would be unacceptable as a TF representation of this signal. Overlapping STFT could be used for better signal tracking, without any influence to the resolution.

A way to improve TF representation of this signal is in transforming the signal into a sinusoid whose constant frequency is equal to the IF value of the LFM signal at the considered instant. Then, a wide window can be used, with a high frequency resolution. The obtained

result is valid for the considered instant only and the signal transformation procedure should be repeated for each instant of interest.

## V. WIGNER DISTRIBUTION

Quadratic TF representations are introduced in order to improve TF concentration of signals with time-varying spectral content and to satisfy some other important properties, like for example, time and frequency marginal properties [2]- [12]. The most important member of quadratic representations is the Wigner distribution (WD). It was defined in the quantum mechanics and later reintroduced in signal analysis by Ville.

A simple way to introduce this distribution is presented. Consider an LFM signal,  $x(t) = A \exp(j\phi(t)) = A \exp(j(at^2/2 + bt + c))$ . Its IF changes in time as  $\Omega_i(t) = d\phi(t)/dt = at + b$ . One of the goals of TF analysis is to obtain a function that will fully concentrate the signal power along its IF. In this case an ideal representation would be  $2\pi A^2 \delta(\Omega - \Omega_i(t))$ . For a quadratic function  $\phi(t)$ , it is known that

$$\begin{aligned} \tau \frac{d\phi(t)}{dt} &= \phi\left(t + \frac{\tau}{2}\right) - \phi\left(t - \frac{\tau}{2}\right) \\ &= \tau(at + b) = \tau\Omega_i(t). \end{aligned}$$

This property can easily be converted into an ideal TF representation for an LFM signal by using

$$\begin{aligned} \text{FT}_{\tau}\{x(t + \tau/2)x^*(t - \tau/2)\} &= \\ \text{FT}_{\tau}\{A^2 e^{j\Omega_i(t)\tau}\} &= 2\pi A^2 \delta(\Omega - \Omega_i(t)). \end{aligned}$$

The FT of  $x(t + \tau/2)x^*(t - \tau/2)$  over  $\tau$ , for a given  $t$ , is the WD. Its definition, in a pseudo form (including window), is

$$\begin{aligned} \text{WD}(t, \Omega) &= \int_{-\infty}^{\infty} x\left(t + \frac{\tau}{2}\right)x^*\left(t - \frac{\tau}{2}\right) \\ &\quad \times w\left(\frac{\tau}{2}\right)w\left(-\frac{\tau}{2}\right)e^{-j\Omega\tau}d\tau. \end{aligned} \quad (5)$$

Soon after it was introduced in signal processing it has been concluded that, due to its quadratic nature, this distribution has very emphatic cross-terms, limiting its applications with multicomponent signals. The cross-terms correspond to the product of one signal component in  $x(t + \tau/2)$  with other component in  $x^*(t - \tau/2)$ . The main research direction for decades was to attenuate the

cross-terms once the WD or its two-dimensional FT (well known ambiguity function (AF)) is calculated. Various forms of two-dimensional smoothing of the WD are proposed, using the property that the cross-terms are oscillatory in the TF domain (Fig. 3(b) and *Example 4*). After the WD or the AF is calculated, two-dimensional low pass kernels are used to suppress the cross-terms (Choi-Williams, Butterworth, Sinc, optimal Gaussian, Zao-Atlas-Marks... kernels [2]- [6]). Keeping the values of the AF along the axes unchanged, the marginal properties are preserved. The reason for introducing so many distributions lies in the fact that the cross-terms reduction and high concentration of auto-terms are two contradictory requirements. In order to deal with this issue, various compromises were made, resulting in various distributions.

## VI. FROM THE STFT TO THE WD

A simple way to prevent the cross-terms appearance while gradually transforming TF representation from the cross-terms free STFT (its squared modulus - spectrogram) toward the WD is based on the relation between two the most important TF representations, the STFT and the WD, established via the S-method (SM), [12],

$$SM(t, \Omega) =$$

$$\frac{1}{\pi} \int_{-\infty}^{\infty} P(\theta)S(t, \Omega + \theta)S^*(t, \Omega - \theta)d\theta.$$

This relation easily follows from (5), replacing  $x(t + \tau/2)w(\tau/2)$  by its inverse  $\int_{-\infty}^{\infty} S(t, \theta) \exp(j\theta\tau/2)d\theta$  from (1). The special cases of the SM are the WD for  $P(\theta) = 1$  and the spectrogram for  $P(\theta) = \pi\delta(\theta)$ . By increasing the width of  $P(\theta)$  from the spectrogram case, we get a gradual transition toward the highly concentrated WD. The best choice of the width of  $P(\theta)$  is the one which enables complete integration over the auto-terms, without the cross terms. Then, the SM produces a sum of the WDs of signal components.

A discrete form of the SM reads

$$SM_L(n, k) = \sum_{i=-L}^L S_N(n, k + i)S_N^*(n, k - i)$$

for  $P(i) = 1$ ,  $-L \leq i \leq L$  (a weighted form  $P(i) = 1/(2L + 1)$  could be used).

A recursive relation for the SM calculation is

$$\begin{aligned} SM_L(n, k) &= SM_{L-1}(n, k) \\ &\quad + 2\text{Re}[S_N(n, k + L)S_N^*(n, k - L)], \end{aligned}$$

The spectrogram is the initial distribution  $SM_0(n, k) = |S_N(n, k)|^2$  and  $2\text{Re}[S_N(n, k + i)S_N^*(n, k - i)]$ ,  $i = 1, 2, \dots, L$  are the correction terms. Considering the parameter  $L$  as a frame index, we can make a video of the transition from the spectrogram to the WD.

There are two ways to implement summation in the SM. The first one is with a constant  $L$ . Theoretically, in order to get the WD for each individual component, the number of correcting terms  $L$  should be such that  $2L$  is equal to the width of the widest auto-term. This will guarantee cross-terms free distribution for all components which are at least  $2L$  frequency samples apart. The optimal number of correction terms can be found by measuring the SM concentration (sparsity), as a function of  $L$ , using the norm-one of the STFT or norm-one-half of the spectrogram and the SM, as in [8] (see *Example 3*).

The second way to implement the SM is with a TF dependent  $L = L(n, k)$ . The summation, for each point  $(n, k)$ , is performed as long as the absolute values of  $S_N(n, k + i)$  and  $S_N^*(n, k - i)$  for that  $(n, k)$  are above an assumed reference level (established, for example, as a few percents of the STFT maximal value). Here, we start with the spectrogram,  $L = 0$ . Consider the correction term  $S_N(n, k + i)S_N^*(n, k - i)$  with  $i = 1$ . If the STFT values are above the reference level then it is included in summation. The next term, with  $i = 2$  is considered in the same way, and so on. The summation is stopped when a STFT in a correcting term is below the reference level. This procedure will guarantee cross-terms free distribution for components that do not overlap in the STFT.

Note that, the SM calculation is alias-free for any  $L$  (including the alias-free WD), with the same signal sampling rate as in the STFT (*Examples 3 and 4*).

## VII. GENERALIZATIONS

The smoothed spectrogram is composed of two STFTs  $S_N(n, k + i)S_N^*(n, k + i)$  for several values of

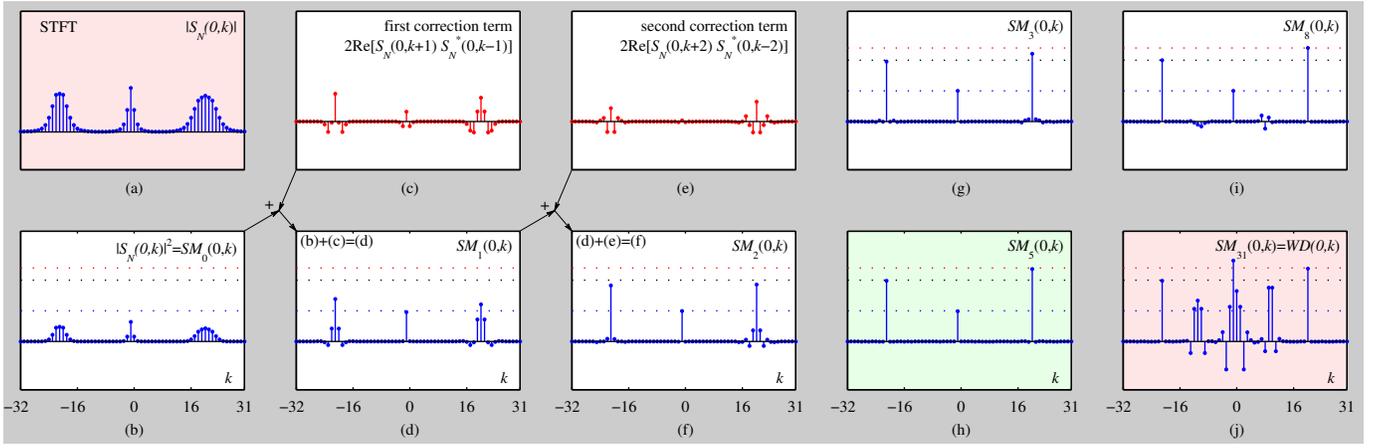


Fig. 2. Analysis of a signal consisting of three LFM components (at the instant  $n = 0$ ). (a) The STFT with a cosine window of the width  $N = 64$ . (b) The spectrogram. (c) The first correction term. (d) The SM with one correction term. (e) The second correction term. (f) The SM with two correction terms. (g) The SM with three correction terms. (h) The SM with five correction terms. (i) The SM with eight correction terms. (j) The WD (the SM with  $L = 31$  correction term).

$i = 0, \pm 1, \dots$  in the same direction of index  $i$ , resulting in the distribution spread, in contrast to the SM, where two STFTs are composed in a counter-direction,  $S_N(n, k+i)S_N^*(n, k-i)$ . These forms were used as a basis to divide the estimators of discrete time-varying processes into two classes in [7].

Considering the SM as a concept of composing two transforms, it can be generalized and used in the realization of cross-terms free higher order TF representations (like polynomial Wigner-Ville, L-Wigner or complex-time distributions), starting from the STFT. Other forms of the SM are also studied, like a time-direction form, a two-dimensional TF form, a fractional or local polynomial FT form, and an affine form. The SM has also been used as a tool for an efficient signal decomposition, [12]. The SM calculation based on TF-varying windows in the STFT is used for a TF representation of noisy signals [9].

## VIII. CONCLUSION

The STFT as a time and frequency localized version of the Fourier transform is presented. It has been shown that this representation can be gradually transformed into better concentrated Wigner distribution. From this transition process we can learn about: the auto-terms concentration improvement, the cross-terms appearance, how to control them, and how to obtain a representation combining good properties of the cross-terms free STFT and highly concentrated Wigner distribution.

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**Example 1:** Consider a signal  $x(n)$  with  $M = 16$  samples, whose values are  $\mathbf{x} = [0.5, 0.5, -0.25, j0.25, 0.25, -j0.25, -0.25, 0.25, -0.25, 0.25, 0.5,$

$0.5, -j0.5, j0.5, 0, -1]$ . Some of its *nonoverlapping STFTs* are calculated according to (2) and shown in Fig.1. Different representations can be compared based on the concentration measures, for example,

$$\mu[S_N(n, k)] = \sum_n \sum_k |S_N(n, k)| = \|\mathbf{S}\|_1.$$

The best STFT representation, in this sense, would be the one with the smallest  $\mu[S_N(n, k)]$ , [8]. For the considered signal and its four representations shown in Fig.1 the best representation, according to this criterion, is the one shown in Fig.1(b). If we know the best concentrated STFT representation of signal we may use it to define an efficient filter form using TF mask  $B(n, k) = 1$  for the TF part of plane with significant signal values and  $B(n, k) = 0$  for noise-only parts. When the measure is used for concentration comparison of different representations it is recommended to use their normalized values.

**Example 2:** A nonoverlapping STFT is presented in Fig.1(a). Calculation instants are  $n_i \in \{2, 6, 12\} = \mathbf{N}_C$ . If we want to use only this STFT for a TF representation of a signal, then the STFT values calculated at  $n_i \in \mathbf{N}_C$  should be used for signal parameters estimation at instants  $n_i \in \{0, 1, 3, 4, 5, 7, 8, 9, 10, 11, 13, 14, 15\}$ . This could be too rough for many applications. Then, instead of using the STFTs calculated for  $n_i \in \mathbf{N}_C$  we would like to calculate STFT for some or all  $n_i \notin \mathbf{N}_C$ . For example,

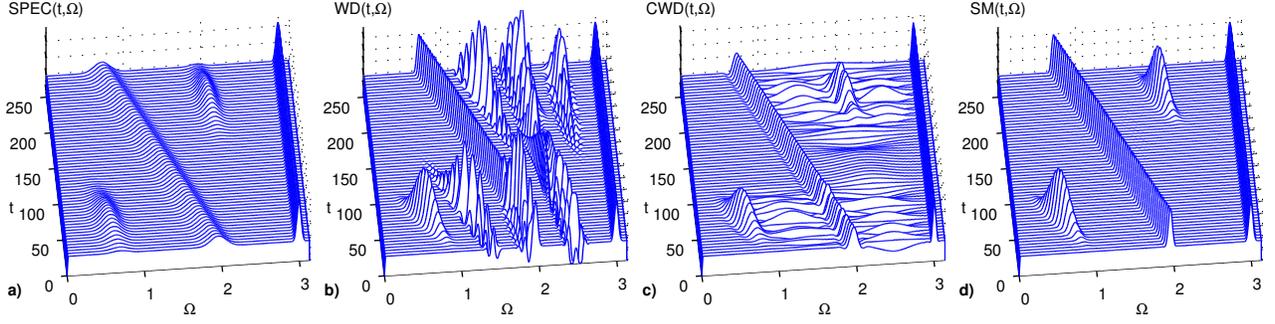


Fig. 3. Time-frequency representation of a four component signal: (a) the spectrogram, (b) the Wigner distribution, (c) the Choi-Williams distribution, and (d) the S-method.

we may also want to calculate STFTs at  $n_i \in \{1, 4, 8, 11, 13, 14, 15\}$ , with respective window widths  $N_i = 2, 4, 4, 2, 2, 1, 1$  (Fig.1(b)). The overlapped STFT, calculated at  $n_i \in \{1, 2, 4, 6, 8, 11, 12, 13, 14, 15\}$ , can be written in a matrix form (4) by using appropriate matrices  $\mathbf{W}_{N_i}$  and  $\mathbf{H}_{N_i}$  with overlapping in rows, corresponding to the signal overlapping in time. In this case, there are  $2M$  STFT values calculated for a signal with  $M$  samples. These STFT values are not independent. Dimension of the transformation matrix  $\tilde{\mathbf{W}}$  in (4) is now  $2M \times M$ . Another possible way of writing an overlapping STFT is in splitting it into nonoverlapping STFTs (as in the STFTs presented in Fig.1(a) and Fig.1(b) and denoted by  $\mathbf{S}^a$  and  $\mathbf{S}^b$ , respectively, using the corresponding figure labels as a superscript). The full STFT set  $\mathbf{S}$ , with overlapping  $\mathbf{S}^a$  and  $\mathbf{S}^b$ , is calculated based on (3) as

$$\begin{bmatrix} \mathbf{S}^a \\ \mathbf{S}^b \end{bmatrix} = \begin{bmatrix} \tilde{\mathbf{W}}^a \\ \tilde{\mathbf{W}}^b \end{bmatrix} \mathbf{x}.$$

For the analysis of signal inversion, by using the overlap and add method, we can write  $\mathbf{S}^a + \mathbf{S}^b = (\tilde{\mathbf{W}}^a + \tilde{\mathbf{W}}^b)\mathbf{x}$ .

For instants  $n = 0, 3, 5, 7, 9, 10$  we can use one of the existing STFTs that include these instants in calculation or calculate new STFTs, introducing the third, fourth,... overlapping layer.

A special case of overlapping STFT with a constant rectangular window and step 1,  $n_i = n_{i-1} + 1$ , can be calculated from (2) in a recursive way

$$S_N(n, k) = e^{j2\pi k/N} \{S_N(n-1, k) + (-1)^k [x(n + \frac{N}{2} - 1) - x(n - \frac{N}{2} - 1)]\}.$$

For the Hann(ing) window, the STFT is related to the STFT calculated with a rectangular window as  $S_N^H(n, k) = \frac{1}{2}S_N(n, k) + \frac{1}{4}S_N(n, k-1) + \frac{1}{4}S_N(n, k+1)$ . Similar relations may be written for the Hamming and the Blackman window.

**Example 3:** A signal consisting of three LFM components,  $x(n) = \sum_{i=1}^3 A_i \exp(ja_i \pi n / 32 + jb_i \pi n^2 / 1024)$ , with  $(a_1, a_2, a_3) = (-21, -1, 20)$  and  $(b_1, b_2, b_3) = (2, -0.75, -2.8)$ , is considered at the instant  $n = 0$ . The IFs of the signal components are  $k_i = a_i$ , while the normalized squared amplitudes of the components are indicated by dotted lines in Fig.2. An ideal TF representation of this signal, at  $n = 0$ , would be  $I(0, k) = A_1^2 \delta(k - k_1) + A_2^2 \delta(k - k_2) + A_3^2 \delta(k - k_3)$ . The starting STFT, with the corresponding spectrogram, obtained by using the cosine window of the width  $N = 64$  is shown in Fig.2(a),(b). The first correction term is presented in Fig.2(c). The result of summing the spectrogram with the first correction term is the SM with  $L = 1$ , Fig.2(d). The second correction term (Fig.2(e)) when added to  $SM_1(0, k)$ , produces the SM with  $L = 2$ , Fig.2(f). The SMs for  $L = 3, 5$ , and  $8$ , ending with the WD ( $L = 31$ ) are presented in Fig.2(g)-(j). Just a few correction terms are sufficient in this case to achieve a high concentration. The cross-terms start appearing at  $L = 8$  and increase as  $L$  increases toward the WD. They make the WD almost useless, since they cover a great part of the frequency range, including some signal components (Fig.2(j) and Example 4).

The optimal number of correction terms  $L$  is the one that produces the

best SM concentration (sparsity), using the norm-one-half of the spectrogram and the SM (corresponding to the norm-one of the STFT), [8], Example 1. In this case the best concentrated SM is detected for  $L = 5$ .

**Example 4:** A four component real-valued signal with  $M = 384$  samples is considered. Its STFT is calculated with a Hann(ing) window of the width  $N = 128$  with a step of 4 samples. The spectrogram ( $L = 0$ ) is shown in Fig.3(a). The alias-free WD ( $L = N/2$ ) is presented in Fig. 3(b). The Choi-Williams distribution of analytic signal is shown in Fig. 3(c). Its cross-terms are smoothed by the kernel, that also spreads the auto-term of the LFM signal and chirps. The SM with  $L = 10$  is shown in Fig. 3(d). For graphical presentation, the distributions are interpolated by a factor of 2. In all cases the pure sinusoidal signal is well concentrated. In the WD and the SM the same concentration is achieved for the LFM signal.

If the STFT matrix is rewritten as  $\mathbf{S} = [\mathbf{S}_N(n_0), \dots, \mathbf{S}_N(n_k)]$ , with rows corresponding to frequency and columns to time, then the SM, for a given  $L$ , can be implemented as a MATLAB function

```
function SM=SM_calc(S,L)
N=size(S,1);
SM=abs(S).^2;
for k=1:L;
    SM(1+k:N-k,:) = SM(1+k:N-k,:) + ...
        2*real(S(1:N-2*k,:)).*...
        conj(S(1+2*k:N,:));
end
%MATLAB codes at www.tfsa.ac.me/LN
```

A simple SM calculation form, through the STFT corrections, along with the recursive STFT realization (Example 2) is a basis for online SM realization.