Time-Frequency Plane Splitting for the First Order Local Polynomial Fourier Transform

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Abstract — **In this paper we present a time-frequency plane tiling (splitting) approach for the local polynomial Fourier transform. Comparison of the proposed approach with the one based on the short-time Fourier transform is given. Advantages of the first order local polynomial Fourier transform in the localization and analysis of LFM signals are shown. Signals that can locally be approximated by the LFM signals are also considered. Theory is illustrated by several examples.**

Keywords — **Time-frequency tilling, LFM signals, LPFT, signal processing, STFT, Time-frequency signal analysis**

I. INTRODUCTION

MANY techniques and time-frequency (TF) tools have been developed during the past years in order to process signals with time-varying spectral content. The short-time Fourier transform (STFT), as a direct extension of the Fourier transform, is one of the basic TF representations. It is used in many applications due to its simplicity and linearity [1]-[4]. The instantaneous frequency is one of the central terms in the time-frequency signal analysis. For a general form of a signal, denoted by:

$$x(t) = A(t)e^{j\phi(t)}, \tag{1}$$

with slow variations of amplitude, comparing to the phase variations, it is defined as the first derivative of signal's phase, i.e. $\omega(t) = \phi'(t)$. The main STFT drawback is in its low concentration for signals with significant instantaneous frequency variations within a window [1]-[4].

In order to overcome this STFT disadvantage, numerous TF representations have been proposed. One of them is the local polynomial Fourier transform (LPFT). This transform is also a linear time-frequency representation with respect to the signal. Due to the additional parameters it can compensate signal variations and produce a highly concentrated representation for the signals which can be considered as polynomial phase signals within the analysis window [1], [4]-[9].

Time-frequency grid tiling by using various window forms in the STFT, including time-varying, frequency-varying or time-frequency varying windows, is considered in time-frequency analysis in order to adjust the window form to the signal behavior, [1]-[4].

In this paper we present a time-frequency grid splitting by using time-varying window in the LPFT.

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After introducing the basic theory concepts of the TF plane splitting for the STFT in Section 2, a discussion of this approach in the LPFT case is given in Section 3. Examples and conclusions are given in Sections 4 and 5.

II. BASIC THEORY

The basic idea behind the STFT is to apply the Fourier transform to a portion of the original signal, obtained by introducing a sliding window function $w(t)$ which will localize, truncate (and weight) the analyzed signal $x(t)$. The Fourier Transform is calculated for the localized part of the signal. It produces the spectral content of the portion of the analyzed signal within the time interval defined by the width of the window function [1]-[4]. The STFT is then obtained by sliding the window along the signal.

Analytic formulation of the STFT is:

$$STFT(t,\Omega) = \int_{-\infty}^{\infty} x(t+\tau)w(\tau)e^{-j\Omega\tau}d\tau\ . \tag{2}$$

Discrete formulation of this representation is given by:

$$S_N(n,k) = \sum_{m=-N/2}^{N/2} x(n+m)w(m)e^{-j\frac{2\pi}{N}mk}\ . \tag{3}$$

The STFT calculated by using equation (3) uses signal samples from the interval $[n-N/2,\ n+N/2-1]$, where N denotes the window width. The window width may be constant or varying, for different time instants. In a matrix form, equation (3) can be written as [4]:

$$\mathbf{S}_{N_i}(n_i) = \mathbf{W}_{N_i}\mathbf{x}_{N_i}(n_i), \tag{4}$$

where a rectangular window of width N_i is used for time instants n_i. In equation (4) $\mathbf{S}_{N_i}(n_i)$ represents a column vector of elements $S_{N_i}(n_i,k)$, $k=-N_i/2,...,N_i/2-1$, while $\mathbf{x}_{N_i}(n_i)$ is a column vector consisted of signal samples $x(n_i+m)$, $m=-N_i/2,...,N_i/2-1$. The notation $\mathbf{W}_{N_i}$ is used for a DFT matrix whose elements are $\exp(-j2\pi mk/N_i)$.

In the case of time-varying window, where the window width N_i changes in time instants n_i, $i=0,1,...K$, the non-overlapping STFT has a form [4]:

$$\mathbf{S} = \tilde{\mathbf{W}}\mathbf{x}, \tag{5}$$

where $\tilde{\mathbf{W}}$ is a matrix:

$$\tilde{\mathbf{W}} = \begin{bmatrix} \mathbf{W}_{N_0} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{W}_{N_1} & \dots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{W}_{N_k} \end{bmatrix}. \quad (6)$$

The vector $\mathbf{x} = [x(0), x(1), ..., x(M-1)]^T$ includes all signal samples, and the column vector $\mathbf{S}$ includes vectors $\mathbf{S}_{N_i}(n_i)$ given by (4), for all i. One coefficient $S_{N_i}(n_i, k)$ in the vector $\mathbf{S}_{N_i}(n_i)$ is a block in the time frequency plane, which covers N_i signal samples from the interval $[n_i - N_i/2,\, n_i + N_i/2 - 1]$, calculated by using a rectangular window centered at time instant n_i. In the frequency direction, block is positioned at $2\pi k / N_i$, while it covers M / N_i DFT samples. Number of the representation's coefficients is the same as the number of the signal samples in this case. Relations (4) and (5) give a formal mathematical background for various time-frequency plane splitting approaches by using the time-varying windows [1]-[4].

For signals with a polynomial phase, the polynomial Fourier transform can be used in order to achieve a high concentration in the frequency domain [5]-[8]. If a nonstationary signal can be considered as a polynomial phase signal within the analysis window, the LPFT can be used in order to achieve a higher concentration [3],[5]-[8]. The LPFT can be considered as a higher-order generalization of the STFT. As in the STFT, the LPFT is a linear time-frequency representation which does not produce cross-terms when $x(t)$ is a multicomponent signal. The *M*-th order LPFT is defined by [3],[5]-[8]:

$$LPFT(t, \Omega; \vec{\Omega}) = \int_{-\infty}^{\infty} x(t+\tau) w(\tau) e^{-j\theta(\vec{\Omega}, \tau)} e^{-j\Omega\tau} d\tau, \quad (7)$$

where,

$$\theta(\vec{\Omega}, \tau) = \Omega_1 \frac{\tau^2}{2!} + \Omega_2 \frac{\tau^3}{3!} + ... + \Omega_M \frac{\tau^{M+1}}{(M+1)!} \quad (8)$$

is $(\mathrm{M}+1)$-th order polynomial with variable τ and $\vec{\Omega} = (\Omega_1, \Omega_2, ..., \Omega_M)$ is the polynomial coefficient vector. The number of coefficients in the exponent determines the order of the representation. The zero order LPFT is equal to the STFT. It is important to emphasize that in our analysis and examples, the LPFT of first order will be used. Its analytic form is:

$$LPFT(t, \Omega; \Omega_1) = \int_{-\infty}^{\infty} x(t+\tau) \omega(\tau) e^{-j\Omega_1 \frac{\tau^2}{2}} e^{-j\Omega\tau} d\tau, \quad (9)$$

where Ω_1 represents a coefficient in modulation part of the representation. A discrete form of the *M*-th order LPFT is:

$$LPFT(n,k) = \sum_{n=-N/2}^{N/2-1} x(n+m) w(m) e^{-j\sum_{i=1}^{M} \omega_i \frac{m^{(i+1)}}{(i+1)!}} e^{-j\frac{2\pi}{N}nk}. \quad (10)$$

In order to calculate (10) by using the fast Fourier transform algorithms, this discrete form can be written as:

$$LPFT(n,k) = DFT\left[x(n+m) w(m) e^{-j\sum_{i=1}^{M} \omega_i \frac{m^{i+1}}{(i+1)!}} \right], \quad (11)$$

where $DFT[\cdot]$ is a Fourier transform operator.

III. TIME-FREQUENCY PLANE SPLITTING FOR THE LPFT

The first order LPFT calculation in the case of LFM signals can be considered as a demodulation of the signal, within the window, with $\exp(-j\Omega_1\tau^2/2)$, and a Fourier transform calculation of the demodulated signal, in order to achieve a concentration of pure sinusoid. It is known that this demodulation is equivalent to a rotation in the time-frequency plane [3],[9]. The coefficient Ω_1 can be directly related with the rotation angle. In the STFT calculation, the rotation angle is zero giving a high concentration in the case of pure sinusoidal signals. It is not the case with LFM signals. The LPFT calculation of the LFM signal can be considered as a STFT calculation of a pure sinusoid, rotated by an appropriate angle. For an arbitrary LFM signal, the first order LPFT, given by (9), is capable of adjusting the parameters in the way that the concentration equivalent to a pure sinusoid in the STFT is achieved.

In order to find the relation between the coefficient Ω_1 and the rotation angle, both axes must have the same units. Discrete forms of the signal and the window, with corresponding STFT and LPFT, will be considered. For a window of length N, it is natural that both time and frequency axes are consisted of N discrete points. Then we observe the LFM signal and calculate how its instantaneous frequency changes within the interval covered by the window, and after that, the instantaneous frequency (which is discrete in the interval of 2π) is scaled on a discrete interval of the length N. By observing changes of the instantaneous frequency, within the window, we came to the conclusion that the tangent of the angle is equal to the ratio of the normalized instantaneous frequency (indexed, because it is transformed on index of DFT) and the window length, i.e.:

$$\tan(\alpha) = \frac{\Omega_1 N}{2\pi} \quad (12)$$

where Ω_1 is the coefficient of the first order LPFT, N is the window length, and α is the rotation angle. Optimal direction can be calculated using the signal moments [3],[9].

The above considerations of the LPFT and the STFT calculations are used to present a generalization of the time-frequency plane splitting for the LPFT case. Theoretical consideration given for relations (4) and (5) can be applied to the LPFT in the same manner. This kind of analysis is equivalent to the windowed fractional Fourier transform approach presented in [9].

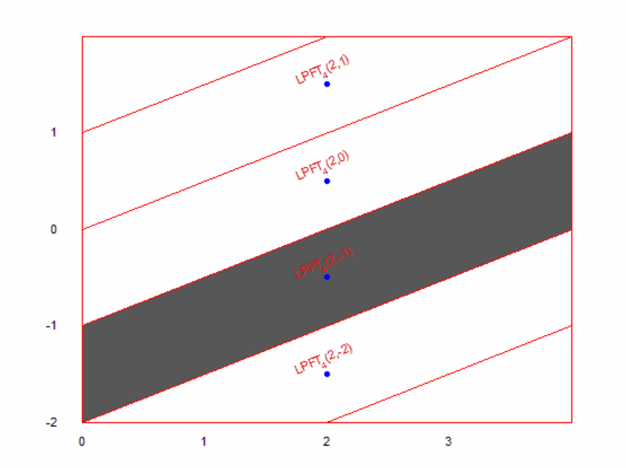

Fig. 1. The LPFT grid for the LFM signal

Figure 1. gives an illustration for an LFM signal in a time-frequency grid of the first order LPFT (or windowed fractional Fourier transform). A rectangular window with fixed size $N=4$ is used for the illustration. The angle given by (12) is the angle between the part of signal (coefficient) which has the most significant value, and the horizontal axis. It refers to a window, which has a rectangular shape just like in the case of the window in STFT. The only difference is in its possibility of rotation, which allows finding a perfect match with the signal, i.e. with the angle (12), which gives the best concentration. The coefficients in this example are calculated by

$$LPFT(2,k) = DFT\left[1,0,0,0\right]e^{j0.6994(-2:1)^2/2} , \qquad (13)$$

where $k \in \{-2,-1,0,1\}$.

IV. Examples

Examples that illustrate the time-frequency plane splitting in the STFT and in the first order LPFT, in the case of LFM signals, are presented in this section. Differences between the STFT and the presented LPFT approach are emphasized. The signals used in these examples are synthetic. Their analytic form is:

$$x = DFT\left[x(1), x(2), ..., x(N)\right]e^{j\Omega_1(-N/2:N/2-1)^2/2} , \qquad (14)$$

where ω_1 represents modulation coefficient, and N is the number of samples.

Example 1. A signal of 128 samples, with five LFM parts appearing in different time intervals, is considered. The window lengths are 32, 16, 32, 16, and 32 respectively. Different window sizes are chosen for different LFM parts of the signal. Two angles different from zero, which have the same values but different signs are obtained by using (12). They are applied for different window lengths and corresponding signal parts. The angles are given as the LPFT coefficients in Table 1.

Two parts of the time-frequency grid where a narrower window of 16 samples is used to illustrate the advantage of the presented LPFT grid over the STFT grid, from the concentration point of view (number of significant values). These grids are shown in Fig. 2.1 and Fig 2.2.

Table 1. LPFT coefficients from Example 1

Signal Part	1	2	3	4	5
Ω_1	0	0.0062	0	-0.0062	0

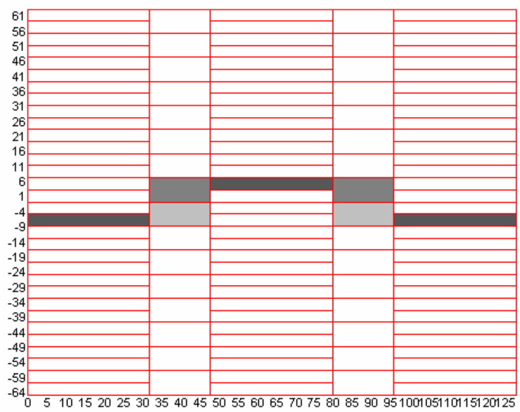

Fig. 2. The STFT grid for the signal from Example 1.

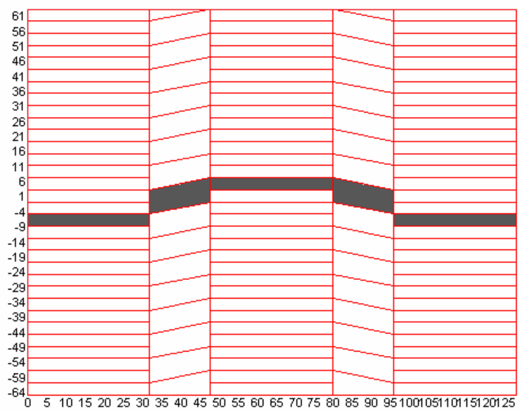

Fig 3. The LPFT grid for the signal from Example 1.

Example 2. A signal of 48 samples is considered. The window lengths used for different time intervals are: 4, 4, 4, 4, 16, 8, 8, respectively. The signal is consisted of three LFM parts (two of them with $\omega_1 \neq 0$). Three different angles are obtained by using (12). The corresponding time-frequency grids are shown in Fig 3.1 and Fig. 3.2. Comparing two different grids, a smaller number of significant coefficients is obtained by using the LPFT approach.

Table 2. LPFT coefficients from Example 2

S. Part	1	2	3	4	5	6	7
Ω_1	4.83	4.1	-4.1	-4.83	0	-0.4354	0

Example 3. Approximation of the time-frequency grid for a sinusoidaly modulated signal, by using the LFM signal coefficients is shown in Fig. 4.1 and 4.2. The sinusoidal FM signal is created by using nine different LFM signals. The resulting signal is consisted of 40 samples.

The window lengths are constant and their value is 8.

Since there are ten LFM parts, ten different angles are used. It is obvious that the STFT can not handle with parts of signal which have LFM nature, as it can the LPFT. For the LFM parts the STFT produced a larger number of significant values than the LPFT. In those parts of signal where the signal's angle (linear modulation coefficient) has a zero value (pure sinusoid), both the STFT and the first order LPFT give the same result, as expected (since the LPFT is a generalisation of the STFT). The LPFT gives the same result (one significant value) for the LFM signals as in the case of pure sinusoids. The approach presented here may be further generalized for any instantaneous frequency law, as described in [10].

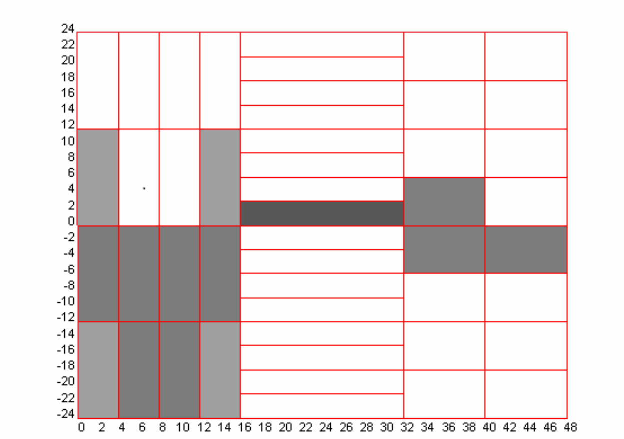

Fig. 4. The STFT grid for the signal from Example 2.

TABLE 3. LPFT COEFFICIENTS FROM EXAMPLE 3

S.P	1	2	3	4	5	6	7	8	9
$\boldsymbol{\Omega_1}$	-3.7	-2.1	0	2.1	3.7	3.7	2.1	0	-3.7

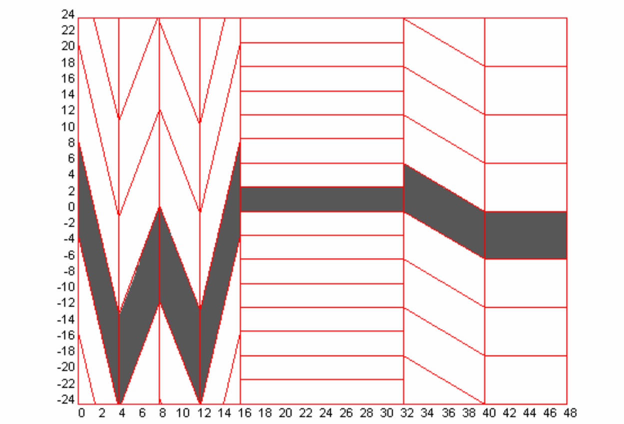

Fig. 5. The LPFT grid for the signal from Example 2.

V. CONCLUSION

This paper extends the time-frequency varying tiling approach from the STFT to the first order LPFT (or windowed fractional Fourier transform) plane. The LPFT window angle is related with the modulation coefficient of a LFM signal, and the LPFT is calculated by using the STFT time-frequency grid tiling approach [1]-[4]. Presentation is supported by appropriate examples, comparing the STFT and the LPFT based results for different signals consisted of time-varying components. In terms of signals with the LFM components, the STFT gives a larger number of nonzero coefficients than the LPFT based analysis. This kind of presentation opens many possibilities for future research, including the analysis of sparse LFM signals, as a potential for the removal of redundant information from representations.

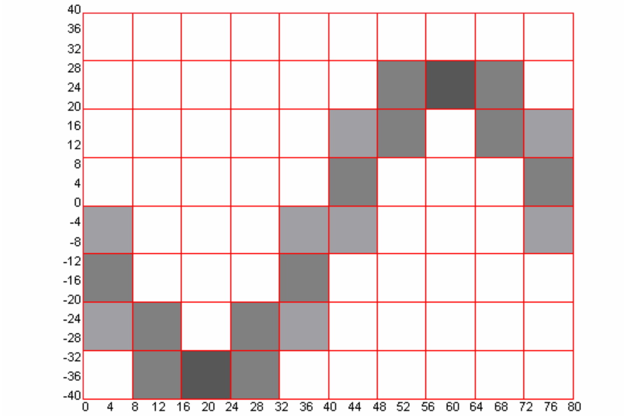

Fig. 6. The STFT grid for Example 3.

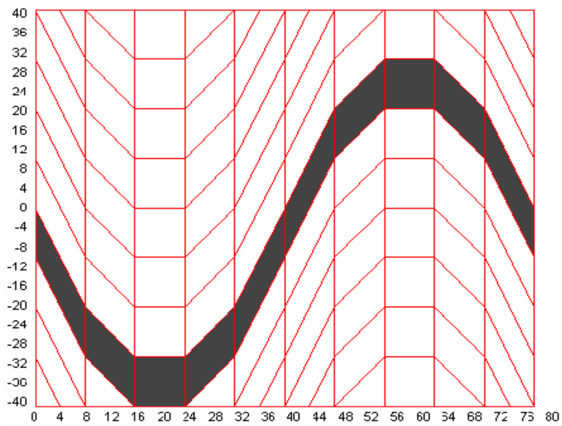

Fig. 7. The LPFT grid for Example 3.

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