Adaptive Gradient Based Algorithm for Complex Sparse Signal Reconstruction

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***Abstract* — An adaptive gradient based algorithm for signal reconstruction from a reduced set of samples is considered in the paper. An extension to complex-valued signals is proposed. It has been assumed that the signals are sparse in a transformation domain. The proposed algorithm is based on the previously published algorithm suitable for real-valued signals only. The algorithm is based on the steepest descent method where the measure of signal sparsity is minimized by varying missing signal samples, using a decreasing step size in iterations. The algorithm performances are analyzed and presented through examples.**

***Keywords* — Compressive sensing, Concentration measure, Signal reconstruction, Sparse signal processing**

I. Introduction

Sparse signal analysis and compressive sensing are emerging areas in signal processing in the last decade [1]-[16]. These areas are closely related since the compressive sensing is possible with the assumption that the considered signal is sparse in a transformation domain. Reconstruction of missing or intentionally omitted signal samples is analyzed in [1]-[12]. The reconstruction process is often formulated as a minimization problem. Compressive sensing is also used in the time-frequency signal analysis [12], [13], biomedical signal processing [14], L-estimation [15], [16], and multimedia signal processing [17].

Consider N samples of discrete complex valued signal $x(n)$ with corresponding discrete Fourier transform (DFT) denoted by $X(k)$. We will assume that signal is sparse with sparsity K in this transformation domain, meaning that only $K < N$ samples of $X(k)$ are non-zero.

Signal sparsity is measured with various sparsity or concentration measures. It is known that the ℓ_0-norm is theoretically the most suitable for counting of the non-zero transform coefficients. The ℓ_0-norm of a sequence $X(k)$ is equal to the number of its non-zero values. However, the minimization with this norm could be done only through a combinatorial search. This search is an NP hard problem. Also, due to the finite calculation precision this measure can not be used in the applications. For example, we can calculate DFT of the properly sampled complex sinusoid and count number of nonzero values in the DFT. Although theoretical result is 1 we will more probably obtain result very close to the number of samples N. In MATLAB/Octave this is implemented with commands

```
n=0:127;
x=exp(j*2*pi*n*10/128);
X=fft(x);
sum(X~=0)
```

We obtain 128 as the output value although when we `plot(abs(X))` it is clear that only one transform value is non-zero. This is a computational reason why other, more robust, norms are used as sparsity measures. The most used one is ℓ_1-norm that is equal to the sum of absolute values of $X(k)$. Sparsity measure is closely related to the concentration measure introduced in [18] where the concentration measure is introduced in time-frequency domain in order to obtain optimal time-frequency signal representation.

Suppose that we have only $M < N$ signal samples, i.e. that some samples are unavailable or intentionally omitted. This scenario belongs to compressive sensing area. In this case it is very important to develop methods for reconstruction of the unavailable signal samples. Advantages of compressive sensing in signal transmission and storage are very important, especially in big data setups.

The reconstruction of missing samples can be formulated as optimization problem, i.e., Minimization of the sparsity measure of the signal transform under condition that available samples remain unchanged. By denoting the measure function with $\mathcal{M}(\cdot)$ and the set of available samples as $\mathbb{N}_x$, the mathematical formulation of the problem is

$$\begin{aligned}&\text{Minimize } \mathcal{M}(\mathrm{DFT}(y(n))) \\ &\text{under constraints } y(n) = x(n) \text{ for } n \in \mathbb{N}_x\end{aligned} \tag{1}$$

where $y(n)$ is the reconstructed signal.

Any problem described with (1) can be solved by a direct search over whole set of possible values. However, such an approach belongs to the class of NP problems that are very time demanding and that can not be applied in real world applications. This is the reason why more efficient algorithms for the solution of (1) are developed.

One of the solutions is to reduce problem to the linear programming (LP) form, and solve it by the primal-dual algorithm [2]. It is implemented in the well known and widely used L1-magic toolbox for signal recovery.

Another approach is recently proposed in [5], [9], and extended to randomly sampled sparse signals in [10], and reconstruction in impulsive noise environments in [11]. It is an iterative procedure based on the adaptive gradient descent algorithm, where a finite approximation of the measure gradient is used. This approach is focused to the

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real-valued signals only. In this paper we will extend this approach to the complex-valued signals.

The paper is organized as follows. Within Section II the reconstruction algorithm is presented. The results obtained by the presented algorithm are given in Section III.

II. Reconstruction Algorithm

The presented algorithm is based on the algorithm for real-valued signal reconstruction proposed in [5] and [9]. The basic idea is to start from the minimum energy solution (all missing samples are set to zero) and to vary missing sample values for $\pm\Delta$ where Δ is appropriately chosen variation step. The measure behavior is checked in order to obtain an estimation of the measure gradient. Next, the missing sample values are adjusted and the whole procedure is repeated. Good starting choice of Δ is signal magnitude

$$\Delta = \max |x(n)|.$$

The values of algorithm step Δ are reduced when algorithm convergence slows down, until the required precision is reached.

In the considered complex-valued signal case a sample variation is done in four directions $\pm\Delta\pm j\Delta$. The estimated gradient vector is complex-valued.

The iterative procedure of the reconstruction algorithm can be summarized as

Step 1: For each missing sample at n_i we form four signals $y_1(n)$, $y_2(n)$, $y_3(n)$, and $y_4(n)$ in each next iteration as:

$$y_1^{(k)}(n) = \begin{cases} y^{(k)}(n) + \Delta & \text{for } n = n_i \\ y^{(k)}(n) & \text{for } n \neq n_i \end{cases}$$

$$y_2^{(k)}(n) = \begin{cases} y^{(k)}(n) - \Delta & \text{for } n = n_i \\ y^{(k)}(n) & \text{for } n \neq n_i \end{cases}$$

$$y_3^{(k)}(n) = \begin{cases} y^{(k)}(n) + j\Delta & \text{for } n = n_i \\ y^{(k)}(n) & \text{for } n \neq n_i \end{cases}$$

$$y_4^{(k)}(n) = \begin{cases} y^{(k)}(n) - j\Delta & \text{for } n = n_i \\ y^{(k)}(n) & \text{for } n \neq n_i \end{cases},$$

where k is the iteration number. Constant Δ is used to determine whether the real and imaginary parts of the considered signal sample should be decreased or increased.

Step 2: Estimate the differences of the signal transform measure as

$$g_r(n_i) = \mathcal{M}\left[\mathrm{DFT}[y_1^{(k)}(n)]\right] - \mathcal{M}\left[\mathrm{DFT}[y_2^{(k)}(n)]\right] \quad (2)$$

$$g_i(n_i) = \mathcal{M}\left[\mathrm{DFT}[y_3^{(k)}(n)]\right] - \mathcal{M}\left[\mathrm{DFT}[y_4^{(k)}(n)]\right] \quad (3)$$

Step 3: Form a gradient vector $\mathbf{G}^{(k)}$ with the same length as the signal $x(n)$. At the positions of the available samples, this vector has value

$$G^{(k)}(n) = 0.$$

At the positions of missing samples its values are

$$G^{(k)}(n_i) = g_r(n_i) + j\, g_i(n_i),$$

calculated by (2) and (3).

Step 4: Correct the values of $y(n)$ iteratively by

$$y^{(k+1)}(n) = y^{(k)}(n) - \frac{1}{N}G^{(k)}(n),$$

Repeating the presented iterative procedure, the missing values will converge to the true signal values, producing the minimal concentration measure in the transformation domain.

Since we use a difference of the measures to estimate the gradient, when we approach to the optimal point, the gradient with norm ℓ_1 will be constant and we will not be able to approach the solution with an arbitrary precision. Instead of moving toward the optimal point we will obtain oscillations, meaning that the gradient vector completely changes direction in subsequent iterations.

This problem may be solved, by reducing the step Δ, for example by $\sqrt{10}$, when we approach the stationary oscillations zone. Oscillations zone can be detected by measuring angle between successive gradient vectors $\mathbf{G}^{(k-1)}$ and $\mathbf{G}^{(k)}$, [10]. Since gradient vectors are complex valued angle between them can be calculated as

$$\beta = \arccos \frac{\Re[\langle \mathbf{G}^{(k-1)}, \mathbf{G}^{(k)} \rangle]}{||\mathbf{G}^{(k-1)}|| \cdot ||\mathbf{G}^{(k)}||} \quad (4)$$

where $\Re$ stands for real part, $\langle \mathbf{G}^{(k-1)}, \mathbf{G}^{(k)} \rangle$ is scalar (dot) product of two complex valued vectors and $||\cdot||$ is vector intensity. Note that for angle calculation we need two successive gradient vectors.

After reduction of the step Δ, the presented procedure is repeated until the required precision is achieved.

Initial value of parameter Δ should be of the signal amplitude order. Good estimate is maximal absolute value of the available signal samples.

Maximal number of iterations should be limited to N_{it} in order to avoid infinite iteration loop in the case when the reconstruction is not possible.

III. Performance analysis

Consider a signal of the form

$$x(n) = \sum_{k=1}^{K} A_k \exp(j\omega_k n + \varphi_k) \quad (5)$$

for $n = 0, 1, \ldots, N-1$. Assume that the discrete frequencies ω_k are on the DFT frequency grid i.e.,

$$\omega_k = \frac{2\pi}{N} m_k$$

where m_k are integers $0 \leq m_1 < m_2 < \ldots < m_K < N$. We will also assume that amplitudes $A_k > 0$. Under this assumptions, the considered signal is sparse in the DFT domain with sparsity K.

We will assume that M randomly positioned signal samples are available and try to reconstruct the remaining $N - M$ samples with the proposed reconstruction algorithm.

A. Reconstruction example

Here we will consider signal of the form (5) with $N = 128$, $K = 3$, $A_1 = A_2 = A_3 = 1$, $\omega_1 = 8\pi/128$, $\omega_2 = 36\pi/128$, $\omega_3 = 240\pi/128$, and $\varphi_1 = \varphi_2 = \varphi_3 = 0$. Number of available samples is $M = 32$.

The considered signal, the reconstruction error and the obtained angles β are presented in Fig. 1. Real and imaginary parts of the original signal are given in Fig. 1(a).

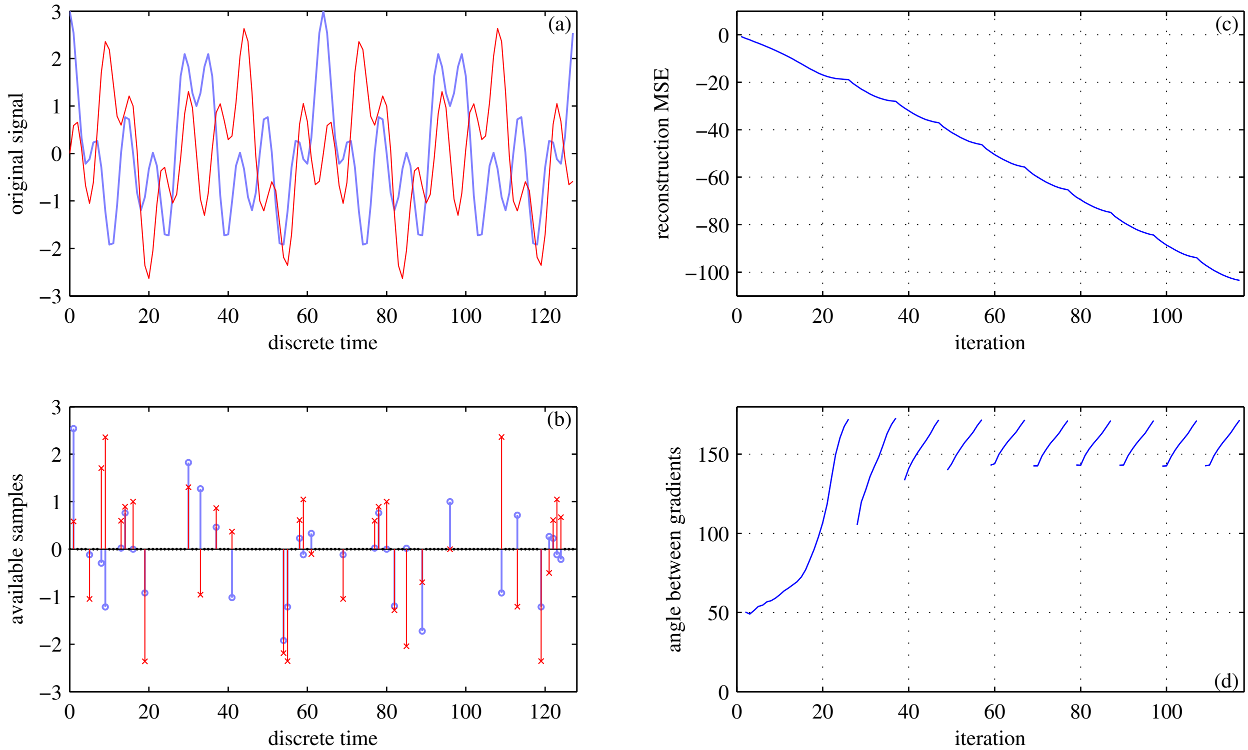

Fig. 1. Reconstruction example: signal length $N = 128$, sparsity $K = 3$ and number of available samples $M = 32$. (a) Real (thick blue line) and imaginary (thin red line) part of the original signal; (b) Available samples real (blue circles) and imaginary (red crosses) parts; (c) Reconstruction MSE calculated for each algorithm iteration; (d) Angle between gradients in successive iterations. Angles ate undefined in the first iteration after Δ reduction.

Available signal samples are plotted in Fig. 1(b). Here $M = 32$ out of $N = 128$ samples are selected.

The reconstruction error is calculated for each iteration and presented in Fig. 1(c), while obtained angles β calculated by (4) are given in Fig. 1(d). From Fig. 1(d) iterations when Δ reduction is performed are clearly visible. Since two successive gradients are required for β calculation, angle β can not be calculated in iteration that immediately succeeds Δ reduction.

The algorithm is stopped after 117 iterations, when the reconstruction error of approximately $-100\,\text{dB}$ is reached.

B. Statistical analysis

Statistical analysis is performed for $N = 128$, $K = 3, 5, 10$ and 20, and $M = 1, 2, \dots, 102$. For each pair (K, M) 100 random realizations of the considered signal (5) are analyzed. Obtained results are summarized in Figures 2-5. Instead of the number of missing samples M we used percentage of the missing samples equal to $100 \times M/N$ as x-axis variable in the figures.

For each realization of the considered signal we have used random discrete frequencies ω_k positioned on the frequency grid, random phases φ_k uniformly distributed over interval from 0 to 2π and the unity amplitudes $A_k = 1$.

Stopping criterion for the reconstruction algorithm is set so that target reconstruction error is approximately $-100\,\text{dB}$.

Figure 2 presents mean value of the reconstruction error. The reconstruction error is calculated per sample and normalized with the signal energy as

$$MSE = 10\log_{10}\frac{\dfrac{1}{N-M}\displaystyle\sum_{n\notin\mathbf{N}_x}|x(n)-y(n)|^2}{\dfrac{1}{N}\displaystyle\sum_{n=1}^{N}|x(n)|^2} \tag{6}$$

where $y(n)$ is the reconstructed signal.

We can see that for each K there is a zone when the reconstruction is not possible. Also there is a transition zone when the reconstruction is possible with some probability. Finally there is a zone where the full reconstruction is obtained in each realization. For a full reconstruction, it is obvious that a higher sparsity requires more signal samples.

Average number of performed iterations for each pair (K, M) is presented in Fig. 3. The highest number of the iterations is required is the transition zone where a full reconstruction is obtained with some probability. Note that the required number of iterations decreases rapidly if we increase the number of available samples. It remains almost constant within the full reconstruction zone.

Probability of a full reconstruction event is shown in Fig. 4. Here we can clearly see the three zones for each sparsity K.

Finally the reconstruction error for all realizations in the case $K = 10$ is presented in Fig. 5. The line on the graph is the mean value over all realizations. The errors obtained in each realization are presented by dots.

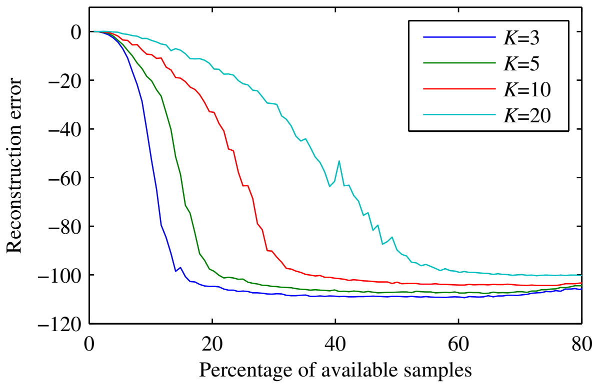

Fig. 2. Reconstruction error as a function of percentage of available samples for signal sparsity $K = 3, 5, 10, 20$

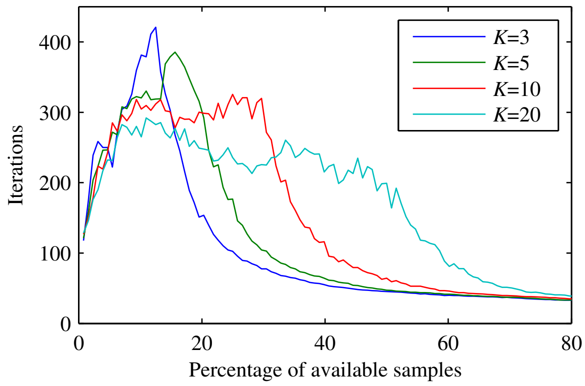

Fig. 3. Average number of iterations as a function of percentage of available samples.

IV. Conclusion

In this paper we have analyzed the reconstruction of missing samples in the case of complex valued sparse signals. New reconstruction algorithm suitable for complex-valued signals is proposed. Algorithm performances are demonstrated on a simple reconstruction case and statistical analysis is performed in order to evaluate algorithm efficiency for various signal sparsity and the number of available samples.

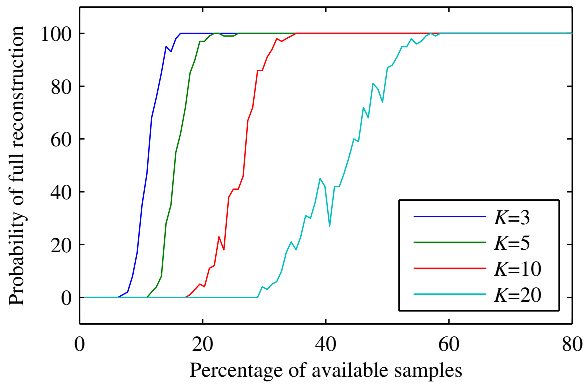

Fig. 4. Probabilty of the full reconstruction event as a function of percentage of available samples

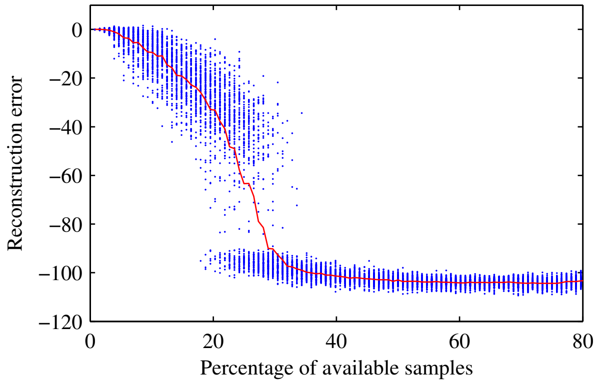

Fig. 5. Reconstruction error as a function of percentage of available samples for 100 realizations (blue dots) and mean value (red line) for sparsity $K = 10$

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