# Robust Hermite transform based on the L-estimate principle

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***Abstract* — Hermite transform has been used in various signal processing applications due to many desirable properties. Particularly, when compared with the trigonometric functions, Hermite functions provide better computational localization in both signal and transform domains. Therefore, this transform has been applied in image compression, tomography, biomedical applications, etc. However, the performance of the Hermite transform may be degraded in the presence of impulse noise, which appears in real applications, for instance, during signal transmission. In such cases, an L-estimation approach is used to mitigate the noise effect. In this paper, the L-estimation approach is applied to the Hermite transform. The goal is to improve signal processing results in the cases of noisy signals for which Hermite transform gives suitable representation. The proposed solution is tested under pure impulse and under combination of the impulse and Gaussian noise, showing improved results in both cases.**

***Keywords* — Gaussian noise, Hermite transform, impulse noise, L-estimation, robust theory.**

## I. Introduction

Robust statistics has been introduced to deal efficiently with signals corrupted by impulse noise. It provides the fundamental principles for solving wide class of problems where impulse disturbances are present in the signal. Basic approaches for signal estimation in impulse, environment, or in a combination of different types of noises, are M-, L- and R- estimate approaches [1]-[7]. In the M-estimate approach, the maximum likelihood estimate is employed for certain class of noise with known statistical characteristics. However, the M-estimate approach is sensitive to the variations of the noise probability density function (pdf), and does not produce closed form solution [4]. Therefore, other approaches are introduced, such as R- and L-estimation. These approaches can provide efficient results even in the case of mixed noises, e.g. Gaussian and impulse type of noise [5]-[7]. Particularly, the L-estimation form is of importance in situations when the noise pdf is unknown. It is based on the principle of the $\alpha$-trimmed filter, which can efficiently deal with signals corrupted by pure noise, impulse or Gaussian, producing median or mean forms as a special cases.

The L-estimation approach can be applied to the discrete signal transforms, in a way to discard certain percent of noise outliers and then to average the rest of the coefficients (signal samples multiplied by basis functions) in order to mitigate the remaining noise [7], [8]. So far, the L-estimation approach has been used mainly with the Fourier basis functions, allowing the definition of the robust Fourier transform, but also the robust time-frequency representations [9]-[14], such as the robust short-time Fourier transform, the robust Wigner distribution and the S-method, robust forms of the complex lag distributions [11] etc.

In this paper we observe the signals in the Hermite transform (HT) domain and propose the L-estimate form of this transformation. The aim is to improve the performance of the standard Hermite transform, which would be highly degraded in the impulse noise environment. The properties of the Hermite basis functions differ from the trigonometric functions, providing in some cases better representation for certain types of signals [15]. The Hermite transform is an orthogonal transformation, used for both 1D and 2D signals, mainly in image coding, computer vision, physics and biomedical applications [16]-[23]. The Hermite functions provide good localization in both signal and transform domain, and thus they can provide compact support of the signals, much better than the other transformations such as DFT, DCT, etc. Further, the Hermite functions allow higher concentration of signal energy at lower frequencies. This property of the Hermite transform is used for signal compression: signal can be approximated with high accuracy by using finite number of Hermite functions for its representation. Having in mind these good properties of the Hermite transform, as well as number of application where this transform has its usage, it is of interest to define robust version of the transform that can efficiently deal with signal corrupted by noise.

The paper is organized as follows: In Section II theoretical background on the Hermite transform is given.

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Section III covers basic theory behind the L-statistics. L-estimate Hermite transform is introduced in this part as well. Experimental results are given in the Section IV, while concluding remarks are in the Section V.

## II. THEORETICAL BACKGROUND

The continuous Hermite functions of order $p$ can be defined as follows [22]:

$$\psi_p(n) = \frac{e^{-n^2} H_p(n)}{\sqrt{2^p p! \sqrt{\pi}}}, \quad (1)$$

where $H_p(n) = (-1)^p e^{n^2} \frac{d^p(e^{-n^2})}{dn^p}$ denotes the $p$-th order Hermite polynomial. As Hermite polynomials satisfy following recursive relation:

$$H_p(n) = 2nH_{p-1}(n) - 2(p-1)H_{p-2}(n), \quad (2)$$

the Hermite functions of order $p$ ($p \geq 2$) can be defined as [17], [22]:

$$\psi_0(n) = \frac{1}{\sqrt[4]{\pi}} e^{-\frac{n^2}{2}}, \quad \psi_1(n) = \frac{\sqrt{2}n}{\sqrt[4]{\pi}} e^{-\frac{n^2}{2}},$$
$$\psi_p(n) = n\sqrt{\frac{2}{p}}\psi_{p-1}(n) - \sqrt{\frac{p-1}{p}}\psi_{p-2}(n), \quad (3)$$

with $H_0(t) = 1$ and $H_1(t) = 2t$. Both, Hermite polynomials and Hermite functions satisfy orthogonality property and, therefore, form an orthogonal basis for signal representation. By using Hermite functions, signal $s(t)$ can be represented as:

$$s(t) = \sum_{p=0}^{\infty} K(p)\psi_p(t). \quad (4)$$

This relation is called the Hermite expansion. The Hermite expansion coefficients are defined as follows:

$$K(p) = \int_{-\infty}^{\infty} s(t)\psi_p(t)dt. \quad (5)$$

In practice, we usually deal with discrete signals of length $N$ and also the finite number $P$ of Hermite functions is used. Therefore, $s(t)$ is usually sampled at the zeros of an $N$-th order Hermite polynomial resulting in a discrete signal $s(n)$. Also, the Hermite functions are calculated at the zeros of $N$-th order Hermite polynomial. The Hermite expansion can be now written as:

$$K(p) = \sum_{n=0}^{N-1} s(n)\psi_p(n), \quad (6)$$

and it represents an approximation of the original signal obtained by using $P$ Hermite functions ($P \leq N$). The coefficients of Hermite transform are usually calculated by using the Gauss-Hermite quadrature approximation:

$$K(p) \approx \frac{1}{N}\sum_{n=0}^{N-1} \mu_{N-1}^{p}(n)s(n), \quad (7)$$

where $\mu$ are constants obtained as:

$$\mu_{N-1}^{p}(n) = \frac{\psi_p(n)}{\left[\psi_{N-1}(n)\right]^2}. \quad (8)$$

The Hermite transform can also be represented in matrix notation:

$$K = \mathbf{H}s,$$

$$\begin{bmatrix} K(0) \\ K(1) \\ \dots \\ K(P-1) \end{bmatrix} = \frac{1}{N} \begin{bmatrix} \frac{\psi_0(0)}{(\psi_{P-1}(0))^2} & \dots & \frac{\psi_0(N-1)}{(\psi_{P-1}(N-1))^2} \\ \frac{\psi_1(0)}{(\psi_{P-1}(0))^2} & \dots & \frac{\psi_1(N-1)}{(\psi_{P-1}(N-1))^2} \\ \dots & \dots & \dots \\ \frac{\psi_{P-1}(0)}{(\psi_{p-1}(0))^2} & \dots & \frac{\psi_{P-1}(N-1)}{(\psi_{p-1}(N-1))^2} \end{bmatrix} \begin{bmatrix} s(0) \\ s(1) \\ \dots \\ s(N-1) \end{bmatrix} \quad (9)$$

where $\boldsymbol{K}$ is the vector of Hermite coefficients, $\boldsymbol{H}$ is the Hermite transform matrix and $\boldsymbol{s}$ is a signal. The inverse formulation of the Hermite transform can be written by using the relations:

$$s = \mathbf{\Psi}K,$$

$$\mathbf{\Psi} = H^{-1} = \begin{bmatrix} \psi_0(1) & \cdots & \psi_0(N) \\ \psi_1(1) & \cdots & \psi_1(N) \\ \vdots & \vdots \ \ddots & \vdots \\ \psi_{P-1}(1) & \cdots & \psi_{P-1}(N) \end{bmatrix}, \quad (10)$$

where $\mathbf{\Psi}$ is the inverse Hermite transform matrix.

## III. L-ESTIMATION APPROACH IN THE HERMITE TRANSFORM DOMAIN

The L-estimation approach, as a part of the robust theory, has attracted significant attention in signal and image filtering [1]-[5]. This approach is used for processing signals corrupted by both Gaussian and impulse type of noise. In this section we will present an L-estimation approach in the Hermite transform domain.

Generally, discrete Hermite transform of the noisy signal can be written as:

$$K(p) = \sum_{n=1}^{N} x(n)\psi_p(n), \quad p \in [1,...,P] \quad (11)$$

where $x(n)=s(n)+v(n)$, $s(n)$ is non-noisy signal and $v(n)$ represents the noise. Filtered signal, i.e. filtered transform domain coefficients $\widehat{K}(p)$ of the noisy signal can be obtained as a solution of the optimization problem, i.e. loss function minimization:

$$\widehat{K}(p) = \arg\min_{\mu} \sum_{n=0}^{N-1} L(e) = \\ = \arg\min_{\mu} \sum_{n=0}^{N-1} L(K(p) - \mu), \quad (12)$$

where $L(e)$ denotes the loss function, and $K(p)$ is the transform of the noisy signal. Based on maximum likelihood (ML) [5] approach, for a given probability density function (pdf), $p_v(e)$, the loss function is determined as:

$$L(e) = -\log[p_v(e)], \quad (13)$$

Mean filter is obtained for $p_v(e) \sim e^{-|e|^2}$, and is used to filter the signals corrupted by Gaussian noise. The loss function in the form $L(e) = |e|^2$ is the ML estimator for such kind of noise. According to the ML approach, in the cases of heavy tailed noise (e.g. impulse, Cauchy or α-stable noise), the loss function should be in the form $L(e) = |e|$. The ML approach is sensitive to the variation of the noise pdf. Also, in practical applications signals are often corrupted by the combination of the impulse and Gaussian noise, rather than by pure Gaussian or pure impulse noise. Therefore, Huber estimation theory provides solution for such situation based on the L-estimation approach.

The L-estimate of the Hermite transform coefficients can be written as [7]:

$$K_L(p) = \sum_{i=0}^{N-1} l_i x_s(i), \quad (14)$$

where $N$ denotes the signal length. The sorted values of the signal multiplied by the Hermite functions are given in vector $x_s$, and it is obtained as follows:

$$x_s(n) = sort\{x(n)\psi_p(n), \;\; n \in (0, N-1)\}. \quad (15)$$

Values of the filter coefficients $l_i$ can be obtained as (for an even $N$):

$$l_i = \begin{cases} \dfrac{1}{4a + N(1-2a)}, & i \in [(N-2)a, a(2-N)+N-1], \\ 0, & otherwise. \end{cases} \quad (16)$$

The filter coefficients are derived in analogy with the α-trimmed mean filter. In such case, the standard form of the Hermite transform is obtained by using $a$=0, while the median form follows for $a$=0.5.

## IV. Experimental Results

### A. *Signal corrupted by impulse noise:*

Consider the signal in the form:

$$x(n) = A_1\psi_{32}(x_m) + A_2\psi_{40}(x_m) + A_3\psi_{60}(x_m) + v(n), \quad (17)$$

where amplitudes $A_1$, $A_2$ and $A_3$ are 0.9, 0.8 and 1.2 respectively, and $v(n)$ is an impulse noise.

Non-noisy signal is shown in Fig. 1a, while Fig. 1b shows corrupted signal. The signal has three components in HT domain, as it can be seen from the Fig. 2a. HT of the noisy signal is shown in Fig. 2b. As it can be seen from the Fig. 2b, the signal components cannot be distinguished from the noisy samples. In order to remove certain amount of noise, the L-estimation approach is used. Signal is firstly multiplied with the Hermite functions and then certain number of noisy coefficients is removed.

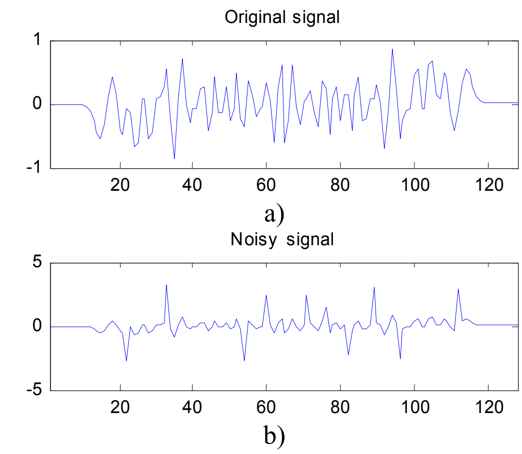


Fig. 1. Time domain of the: a) non-noisy signal; b) signal corrupted by impulse noise

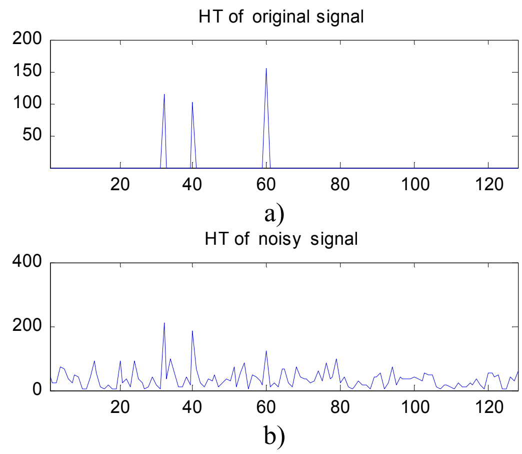


Fig. 2: Hermite transform of the: a) non-noisy signal; b) signal corrupted by impulse noise

In this example approximately 20% of noisy coefficients are removed, and the resulted HT is shown in Fig. 3. As it can be seen, all three signal components are now visible.

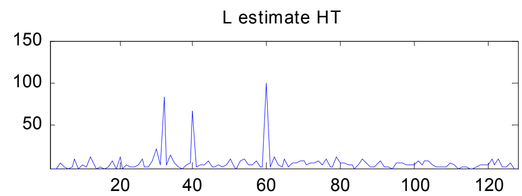


Fig. 3. L-estimate Hermite transform

### B. *Signal corrupted by mixture of Gaussian and impulse noise*

In this example, we consider the same signal from the previous example, but here the signal is corrupted by both, Gaussian and impulse type of noise. Signal to noise ratio

(SNR) is -2.1262 dB. Noisy signal in time and in transform domain is shown in Fig. 4. L-estimation approach has been applied to the coefficients of the signal in transform domain. The same number of coefficients has been removed as in the previous case – 20 % of samples. The obtained HT is shown in Fig. 5, and it can be seen that this approach works well even if both type of noise are present in the signal.

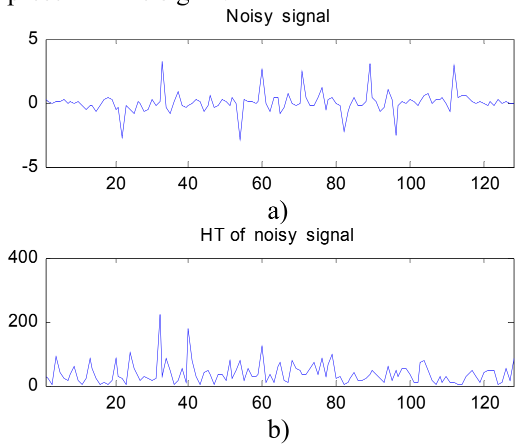


Fig. 4. Signal corrupted by mixture of Gaussian and impulse noise: a) time domain, b) Hermite transform domain

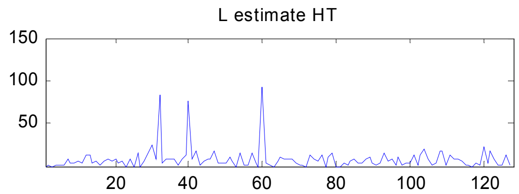


Fig. 5. L-estimate HT of the signal corrupted by Gaussian and impulse noise

## V. Conclusion

The L-estimate form of the Hermite transform has been proposed in the paper. This form of the Hermite transform is shown to be more efficient in dealing with signals corrupted by noise. The impulse noise has been considered as well as mixture of Gaussian and impulse disturbances. Compared to the standard forms of the Hermite transform, it is shown that the L-estimate one can significantly decrease the noise and enhance the signal components.

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