Iterative Denoising of Sparse Images

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***Abstract*—The paper examines an application of the gradient-based algorithm to image denoising with noise values being in the range of the available (non-noisy) pixel values. The analyzed image is considered to be sparse in the 2D-DCT domain. The presented algorithm is a generalization of the previous results on denoising images when the noisy pixels can be detected and eliminated using the L-statistics. The algorithm is based on the recently developed technique used for denoising of one-dimensional signals. A significant advantage of the presented algorithm is that it does not use any a priori knowledge about the positions, values or distribution of the noisy pixels. It is assumed that the positions of noisy pixels cannot be determined using the methods like the L-statistics based ones. Hence, the proposed approach reconstructs the pixels values iteratively using the highest gradient as pixel selection criterion, thus performing blind denoising on a pixel-by-pixel basis. The examples with synthetic two-dimensional signal and a test image are presented. Quality of the image reconstruction is measured using the structural similarity index and the mean absolute error (MAE).**

***Keywords*—compressive sensing, recovery, noise, gradient algorithm, denoising, image processing**

I. Introduction

Compressive sensing (CS) is the field used for the reconstruction of signals which are sparse in a transformation domain. A sparse signal is a signal with very few transform coefficients. Knowing that a signal is sparse, it can be reconstructed using less number of samples than in the conventional way. The main two conditions that have to be met for the successful reconstruction are sparsity and incoherence of the signal. There are many methods and algorithms developed since the introduction of CS, classified into two large groups, one being greedy algorithms and the other based on convex relaxation algorithms. Since many signals are sparse in the transformation domain, the application of the CS algorithms is widely spread in all areas of digital signal processing, such as multimedia, biomedicine, radars, etc. The advantage in using CS algorithms is that it is memory efficient and, as long as the conditions are met, it gives a unique solution to the reconstruction [1]-[7]. The algorithm used in this paper is a modified version of the gradient-based algoritjm [8], [9], from the convex relaxation group. The algorithm showed good results in image denoising in the presence of different kind of noise. The advantage of this algorithm is that the image does not have to be strictly sparse. So far, the cases of salt and pepper and Gaussian noise were considered. In this paper, we use the uniform random noise, which means that the noise values are within the range of the uncorrupted pixels. That is, the detection is different and much harder in this case than in the previous ones. Another complex circumstance is that we do not have the knowledge of the positions of corrupted pixels.

In this paper, processing of sparse signals will be used in a more specific way. The reduced set of samples will be unchanged and the unavailable samples will be presented as the uniform random noise, so that their values are within the range of the available pixels. In comparison to the previous results of the algorithm, there will be additional steps in the reconstruction. The assumption is that we have missing/corrupted pixels in the spatial domain, and that we know that the original image (i.e. the image with no corrupted pixels) is sparse in the transformation domain. In contrast to the greedy algorithms, the values of the uncorrupted (available) samples will remain unchanged.

The paper is organised as follows: after the introduction part, basic theoretical background about the image reconstruction and proof of the detection of the positions using the algorithm are shown in Section 2. In Section 3, the reconstruction algorithm and its explanation are presented. In Section 4, the numerical results for two-dimensional synthetic signals are shown, as well as for a test image.

II. Theoretical Background

Let us consider an $N \times N$ 8-bit grayscale sparse image with impulsive noise. The image can be defined as

$$x_a^{(0)}(m,n) = \begin{cases} x(m,n), & \text{for } (m,n) \in \mathbb{M} \\ 0 \text{ (or } 255), & \text{elsewhere} \end{cases} \tag{1}$$

where 0 is black and 255 is white colour, with M number of uncorrupted (available) pixels and $\mathbb{M} = \{(m_1,n_1),(m_2,n_2),...,(m_M,n_M)\}$ presents set of the uncorrupted pixels. The image is sparse in the DCT domain and the sparsity is K. For the gradient algorithm, the reconstruction is done in the spatial domain. When the available pixels are known, or the image has impulsive noise so the corrupted ones are distinguishable from the available ones, the image does not have to be strictly sparse.

The approach considered in this paper deals with an image with some number of pixels corrupted with uniform random noise and with a reduced set of available pixels. The aim is to

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reconstruct the pixels that are corrupted without knowing the positions and the number of the corrupted samples. To make the reconstruction possible, the sparsity in the transformation domain should be implicitly assumed.

Before the introduction of the algorithm, we will present why the gradient is high for the unavailable/corrupted samples. Consider a sparse signal $x(m,n)$ as in equation (1). The signal with one corrupted sample at position (m_0,n_0) can be defined as $x_a(m_0,n_0)=x(m_0,n_0)+z$ with z being the uniform noise. Due to the change of z, form the signals

$$\begin{aligned} x_a^{+}(m,n) &= x(m,n)+(z+\Delta)\,\delta(m-m_0,n-n_0)\\ x_a^{-}(m,n) &= x(m,n)+(z-\Delta)\,\delta(m-m_0,n-n_0) \end{aligned} \tag{2}$$

where Δ is the gradient parameter. The sparisty measure is defined as

$$g(m_0,n_0)=\frac{\|\mathbf{X}_a^{+}\|_1+\|\mathbf{X}_a^{-}\|_1}{2N\Delta} \tag{3}$$

where $\mathbf{X}_a^{+}$ and $\mathbf{X}_a^{-}$ are the vectors of the transformation domains of the signals in (2). Let us consider that the transformation domain is the two-dimensional Discrete Cosine Tranform (2D-DCT). The proof for 1D-DFT can be found in [10]. So, the transformation domains of signals $x_a^{+}(m,n)$ and $x_a^{-}(m,n)$ can be written as

$$\begin{aligned} X_a^{+}(k,l) &= X(k,l)+(z+\Delta)\,\varphi(k,l)\\ X_a^{-}(k,l) &= X(k,l)+(z-\Delta)\,\varphi(k,l) \end{aligned} \tag{4}$$

where

$$\varphi(k,l)=\cos\left(\frac{2\pi(2m_0+1)k}{4N}\right)\cos\left(\frac{2\pi(2n_0+1)l}{4N}\right). \tag{5}$$

The 2D-DCT of the known samples is $X(k,l)$ and $(z\pm\Delta)\,\varphi(k,l)$ is the DCT of the missing (corrupted) sample. The measures can be defined as the addition of the measure of the original (non-corrupted samples) and the corrupted one (with the Δ shifts)

$$\begin{aligned} \|\mathbf{X}_a^{+}\|_1 &= \textstyle\sum_{k,l=0}^{N-1}|X_a^{+}(k,l)| \cong \mu+|z+\Delta|\,C\\ \|\mathbf{X}_a^{-}\|_1 &= \textstyle\sum_{k,l=0}^{N-1}|X_a^{-}(k,l)| \cong \mu+|z-\Delta|\,C \end{aligned} \tag{6}$$

where $\mu=\|\mathbf{X}\|_1$ is the sparsity measure of the original signal $x(n)$ and C is a constant dependent on number of samples N and is close to $C\cong\sqrt{\frac{\pi}{2}}$. For example, when $N=64$, the constant $C=1.2214$ and when $N=512$, $C=1.2321$. The sparsity measure can be written as

$$\begin{aligned} g(m_0,n_0)=\frac{\|\mathbf{X}_a^{+}\|_1+\|\mathbf{X}_a^{-}\|_1}{2N\Delta}&\cong\\ \cong\frac{|z+\Delta|\,C-|z-\Delta|\,C}{2N\Delta}&. \end{aligned} \tag{7}$$

For deviations from the true signal value smaller than the step $|z|<\Delta$ we get

$$g(m_0,n_0)\cong\frac{Cz}{\Delta}\sim Cz, \tag{8}$$

which means that the gradient is proportional to the shift of the corrupted samples at its position, and the gradient is equal to zero at the positions of the available samples.

The sparsification is done using the quantization matrix of the JPEG standard. The quantization is applied on the 8×8 DCT blocks with 50% quality factor. In this paper, the standard was assumed with quality factor being 25%, which defines the number of the DCT components considered in a block [11].

III. Reconstruction Algorithm

The gradient algorithm is based on the efficient minimisation of the missing/corrupted samples. For the beginning of the description, assume that the number and positions of the available pixels are known. For this scenario, the one-dimensional algorithm is introduced in [8] and two-dimensional in [12], [13]. Let us consider a two-dimensional signal $x(m,n)$, which is sparse in the 2D-DCT domain. Missing values are considered as variables. Then the signal with only M available samples in the first iteration can be represented as in equation (1).

A. Algorithm

Step 1: Firstly, we add and subtract a value Δ to the corrupted/missing samples. The new signals are

$$\begin{aligned} x_a^{+}(m,n) &= x^{(p)}(m,n)+\Delta\delta(m-m_i,n-n_i)\\ x_a^{-}(m,n) &= x^{(p)}(m,n)-\Delta\delta(m-m_i,n-n_i) \end{aligned} \tag{9}$$

Step 2: The transforms of such signals are calculated and the gradient value is estimated as a finite difference of ℓ_1-norms of these transforms. Based on the gradient value, the missing signal sample $x(m_i,n_i)$ is increased or decreased. For each $(m_i,n_i)\notin\{(m_1,n_1),(m_2,n_2),...,(m_M,n_M)\}$ (i.e. the missing samples) the gradient value is approximately:

$$g(m_i,n_i)=\frac{\|X_a^{+}(k,l)\|_1-\|X_a^{-}(k,l)\|_1}{2\Delta} \tag{10}$$

where $X_a^{+}(k,l)=T\{x_a^{+}(m,n)\}$, $Xa^{-}(k,l)=T\{x_a^{-}(m,n)\}$ and Δ is the missing samples change step. In our case, $T\{.\}$ is the 2D-DCT.

Step 3: The available samples are not changed. Each missing pixel value $(m_i,n_i)\notin\{(m_1,n_1),(m_2,n_2),...,(m_M,n_M)\}$ is changed in the direction opposite of the gradient for a step μ

$$x_a^{(p)}(m_i,n_i)=x_a^{(p-1)}(m_i,n_i)-\mu g(m_i,n_i). \tag{11}$$

Step 4: Since ℓ_1-norm is convex but of specific form, gradient based procedure will approach the solution of the problem up to the value proportional to the steps Δ and μ. When the values are close to the true signal values, they will oscillate around the solution with magnitude proportional to the step. When the oscillation is detected, the step size is reduced, for example as $\Delta=\Delta/3$ and $\mu=\mu/3$. These new parameters continue approach to the true signal values until the new precision is reached. The procedure is repeated until the desired reconstruction accuracy is achieved.

Step 5: The algorithm is stopped when the change in two successive iterations is smaller than the desired accuracy ε,

$$\max_{m,n}\left|x_a^{(p)}(m,n)-x_a^{(p-1)}(m,n)\right|<\varepsilon. \qquad (12)$$

Note that only missing samples contribute to this difference, since the available samples are not changed. The proof of the L_1-norm convexity with respect to the corrupted pixels as variables can be found in [14]. This is the basic reconstruction algorithm with the knowledge of the position of the missing samples or, at least, with very high impulsive noise so that the positions of the corrupted samples/pixels are distinguishable from the non-corrupted ones. In the next subsection, a method for selection of the corrupted pixels is explained.

B. Pixel selection

There are two ways in which the corrupted sample can be selected: single-step and the iterative way. For both ways we repeat Step 1 and 2 (i.e. equations (9) and (10)) for all samples since we do not know the positions of missing ones:

$$x_a^+(m,n)=x^{(p)}(m,n)+\Delta\delta(m-m_i,n-n_i)$$
$$x_a^-(m,n)=x^{(p)}(m,n)-\Delta\delta(m-m_i,n-n_i)$$

and

$$g(m_i,n_i)=\frac{\left\|x_a^+(k,l)\right\|_1-\left\|x_a^-(k,l)\right\|_1}{2\Delta}.$$

Note that the value of Δ here is different to the value of Δ in the algorithm itself. For this calculation, Δ must be a high number i.e. $\Delta>\max_{m,n}=x(m,n)$.

For the single-step estimation, we sort the values of the gradient and set a threshold to find largest M values. The detected positions are considered as positions of the corrupted samples. Other values are considered as the available pixels. A drawback in this calculation is that the information about the number of the missing/corrupted samples (or at least approximate number) is crucial.

For the iterative way, we do not need the information about the number of corrupted samples nor the positions of them. The procedure is similar: each time we take the pixel with the largest gradient, reconstruct it (Note: use the algorithm explained above for only one missing sample with a known position), eliminate it from the array of possible values, and repeat the whole procedure. This will be repeated until the error of two successful iterations is below an acceptable level.

IV. Numerical Results

The algorithm was already tested on one-dimensional signals in [15]. The extension to two-dimensional signals is presented here. In the first subsection, a synthetic two-dimensional signal is shown to explain how the algorithm was improved in the two-dimensional case. In the second subsection, the image reconstruction using a grayscale image ”Lena” is shown.

A. Two-dimensional signals

In this section, a synthetic two-dimensional signal is introduced. The signal is given in the form:

$$x(m,n)=\sum_{i=1}^{K}A_{p_iq_i}\varphi(k_{p_i},k_{q_i}) \qquad (13)$$

where

$$\varphi(k_{p_i},k_{q_i})=\cos\left(\frac{2\pi(2m-1)k_{p_i}}{4N}\right)\cos\left(\frac{2\pi(2n-1)k_{q_i}}{4N}\right).$$

$A_{p_iq_i}$ and k_{p_i,q_i} are random amplitudes and random frequency positions of the signal. The image is a $N\times N$ signal where $N=16$, $M=196$ signal samples are available and the sparsity (i.e. number of DCT components) is $K=6$ (consider 6 components in a $16\times16=256$ samples signal). From this it follows that approximately 25% (196/256) of the image is missing. Fig. 1 shows the result for the case when missing samples are set to zero (and the positions are known). In Fig. 1(top) we present the original image and its DCT, Fig. 1(middle) is the image with the available values (the unavailable are the blue pixels) and Fig. 1(bottom) is the reconstructed image.

The improvement achieved using the introduced algorithm is that we can reconstruct signals whose corrupted samples are within the range of the available pixel values. It is hard to distinguish the difference between the corrupted and available pixels and to find the unavailable ones (as it is seen in Fig. 2). In the first case (explained above), if we do not know the positions of the available ones, the reconstruction will not be successful. In this example, we will take 50% of samples with largest gradient and set them to zero. This will make the algorithm straightforward. The original image, the one with available pixels and the reconstruction image are shown in Fig. 2. The order of images is as in the previous one.

B. Image reconstruction

Unlike the classical reconstruction of the image using the gradient-based algorithm, we need to assure the image is sparse in the DCT domain. We will represent it with the standard JPEG image reconstructed with 8×8 reconstruction blocks.

The 512×512 image “Lena” image is tested using the algorithm presented. As the sparsity should be as high as possible (meaning that we use as few DCT components as possible), without destroying the visual effect too much, it is seen that the quality of image using quality factor $QF=25$ is very similar to the quality of 50% and with less components (i.e. sparser). Because of that, the analysis is done for the $QF=25$ (higher sparsity will give higher probability for reconstruction using CS algorithms). The original image and the sparsified image are shown in Fig. 3. The sparse image is then split into blocks and each block has 12.5% of corrupted samples. The block size is 8×8. The noisy and reconstructed images are shown in Fig. 4.

C. Comparison

The structural similarity (SSIM) index value is defined as a function of luminance, contrast and structure of two images

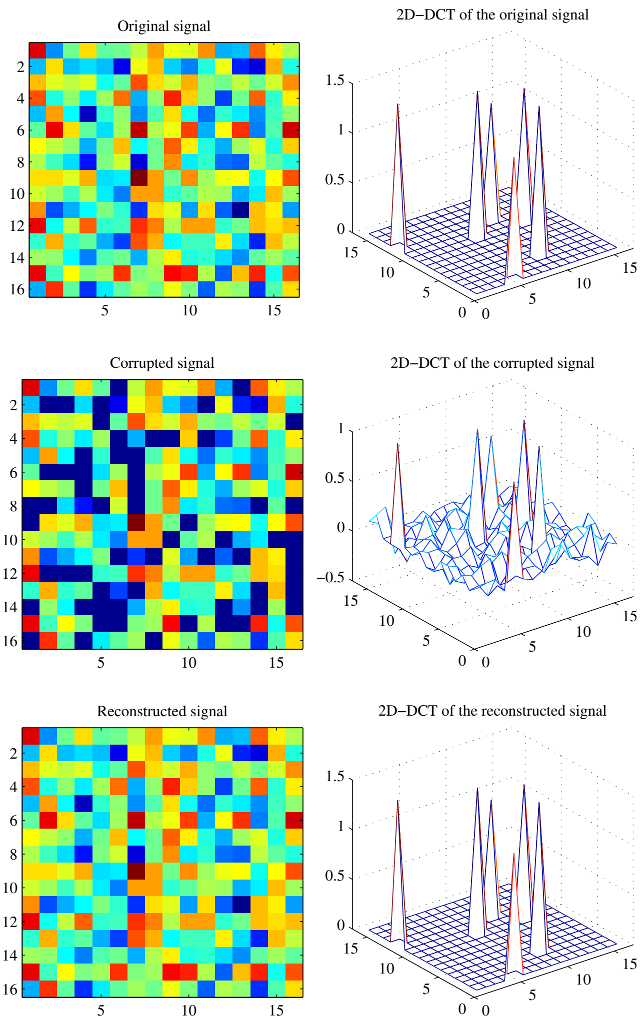

Fig. 1. Original signal and its 2D-DCT (top); Signal with corrupted samples and its 2D-DCT (middle); Reconstructed signal and its 2D-DCT (bottom)

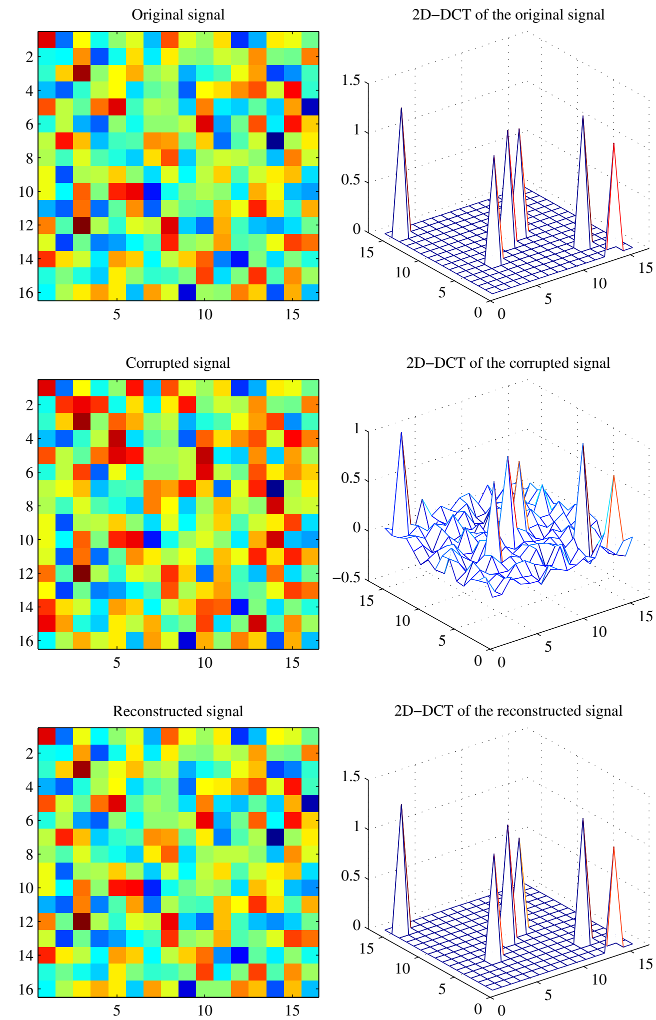

Fig. 2. Original signal and its 2D-DCT (top); Signal with corrupted samples and its 2D-DCT (middle); Reconstructed signal and its 2D-DCT (bottom)

[16], and can be written as

$$\mathrm{SSIM}(x_1, x_2) = \frac{(2\mu_{x_1}\mu_{x_1} + c_1)(2\sigma_{x_1x_2} + c_2)}{(\mu_{x_1}^2 + \mu_{x_2}^2 + c_1)(\sigma_{x_1}^2 + \sigma_{x_2}^2 + c_2)} \quad (14)$$

where x_1, x_2 are the two considered images (in our case x_1 is the original sparse image and x_2 is the reconstructed image), μ_{x_1}, μ_{x_2}, and $\sigma_{x_1}, \sigma_{x_2}$, and $\sigma_{x_1x_2}$ represent the mean, variance and covariance of the two images respectively and c_1, c_2 are stabilization variables. If the SSIM index is close to 1, the images are highly similar, and if the index is close to 0, it means that the images are not similar. In our case, the SSIM index calculated for the reconstructed image is 0.9858. The SSIM index for the median filter of size 3×3 is calculated 0.9825 and for the 5×5 is 0.9486. The MAE is calculated as $MAE(x_1, x_2) = mean(mean(|x_1 - x_2|))$. For the gradient algorithm, it is calculated to be 0.5693, for the median 3×3 is 1.9726, and for 5×5 is 3.3173. The disadvantage of median filtering is that it degrades the whole image. The gradient algorithm reconstructs only the pixels which are found to be corrupted, and the pixels which are set as uncorrupted/available remain unchanged.

V. Conclusions

The gradient-based algorithm for reconstruction of noisy images with noise pixel values being within the range of the available pixel values is shown in this paper. The reconstruction is done on a synthetic two-dimensional image and on a grayscale test image. On the synthetic two-dimensional signals, the reconstruction using the gradient-based algorithm in a conventional way is shown as well, to explain the difference between what was done before and what are the features of the modified algorithm. When the number and the positions of the corrupted samples are unknown (and additionally the corrupted samples are in the range of the available ones), we use the algorithm presented here. It is shown that the reconstruction can be done successfully without the knowledge of corrupted sample positions. There are few parts that we have to consider in the reconstruction of a noisy image. In contrast to the conventional gradient (i.e. when we have the knowledge of the available pixels), the image has to be sparse in the transformation domain.

Fig. 3. Original image (top); Sparse image (bottom)

Fig. 4. Noisy image (top); Reconstructed image (bottom)

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