Representation of Uniformly Sampled Signals in the Hermite Transform Domain

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Abstract **- The problem of the representation of uniformly sampled signals in the Hermite transform basis is revisited. Namely, due to the application of Gauss-Hermite quadrature in the calculation of transformation coefficients, the discrete Hermite transform assumes that the analyzed signal is sampled at the points proportional to the roots of the Hermite polynomial of the corresponding order. Since the most of discrete signals are sampled according to the sampling theorem, there is a particular interest for the representation of such signals. A form of the Hermite transform is defined based on the sampling theorem formula for the reconstruction of continuous-time signals from the samples of their discrete counterparts.**

Keywords – **Digital Signal Processing; ECG signals; Hermite Fucntions; Hermite Transform; QRS complexes**

I. Introduction

The Hermite transform, as an alternative of the Fourier transform, has been studied for decades due to its suitability for the representation of signals with compact time support [1]-[16]. Namely, the specific form of the transform basis functions, advantageous mathematical properties of the representation as well as the fast calculation algorithms lead to numerous applications, including the compression of QRS complexes in ECG signals [1]-[5], image segmentation and edge detection [6], [7], computed tomography, analysis of protein structure [8], radar signals [14], representation and analysis of optical waves [9]. Fast computation algorithms proposed in [2] and [8] ensured the placement of the Hermite transform and Hermite functions in state-of-the-art research in biomedicine and biology [1], [8]. Their good localization properties have found important applications in time-frequency signal analysis, radar signal processing and processing of video signals [16]. Since the Hermite functions are eigen-functions of the Fourier transform, the Hermite transform has been employed in representation and reconstruction of electromagnetic pulse signals [11], [12].

Mainly, the Hermite transform has been introduced in the literature as a series expansion of continuous signals on the basis formed of continuous-time Hermite functions. Several research papers deal with the discrete form of this representation [2]-[4], [6], [9], [10]. The discrete form of the transform assumes that samples are available at the points proportional to the roots of the Hermite polynomial. This fact is very important, since the Hermite coefficients are introduced in an integral form, which has to be properly calculated. Mainly, the Gauss-Hermite quadrature is used for the calculation of the integral, since it forms a complete discrete signal representation if both basis functions and the analyzed signal are sampled at the points proportional to the roots of the Hermite polynomial. If this condition is satisfied, any discrete signal of the length M can be uniquely represented as a summation of adequately scaled M Hermite basis functions [2]-[4], [9], [16].

Most of the reported approaches to the representation of signals in the Hermite domain assume that the analyzed signal is available in its continuous-time form, and that we can sample it at the instants of the interest. However, in practice many analyzed signals have been already sampled uniformly, according to the sampling theorem [16], [17]. The sampling grid corresponding to the roots of the M-th order Hermite polynomial is non-uniform, and the required sampling points are dislocated from the sampling theorem grid. This means that the signal values are not available at instants of the interest, and consequently, the Gauss-Hermite quadrature cannot be applied in the Hermite coefficients calculation.

In this paper, we deal with the problem of the representation of discrete signals in the Hermite domain, assuming that the analyzed signals have been sampled uniformly, according to the sampling theorem. The formula for the reconstruction of continuous-time signal from its samples, which is a consequence of the sampling theorem, will be engaged as the part of the transform. Namely, signal values at the instants proportional to the roots of the Hermite polynomial will be exactly calculated, under the condition that the existing samples are available at the discrete time grid satisfying the sampling theorem [17]. Since the Hermite polynomial roots assume fix predefined values, depending only on the signal length, it will be shown that a scaling factor should be engaged in the resampling process. A discussion on this scaling factor influence is provided. Theoretical contributions are illustrated by a numerical example with real biomedical signal known as QRS complex [1]-[5].

The rest of the paper is organized as follows. Section 2 presents the discrete Hermite transform. The relation between uniformly and non-uniformly sampled signals is discussed in Section 3. Section 4 presents numerical examples illustrating the presented theory, while the concluding remarks are given in the end of the paper.

II. Hermite Transform

Hermite transform enables the signal projection on Hermite basis functions defined as:

$$\psi_p(t,\sigma) = \left(\sigma 2^p p!\sqrt{\pi}\right)^{-1/2} e^{-t^2/2} H_p(t/\sigma), \qquad (1)$$

where the Hermite function order is denoted by p, $H_p(t/\sigma)$ is the p-th order Hermite polynomial, and the scaling factor σ is introduced to "stretch" and "compress" the corresponding Hermite function, thus increasing the basis suitability to better match the signal being represented [2], [4], [10]. The illustration of the first four Hermite functions is given in Fig. 1. It can be easily concluded that these basis functions have a compact time-support. The p-th order Hermite function and the functions of the orders $p-1$ and $p-2$ can be related with the following recursive formula:

$$\psi_0(t,\sigma)=\frac{1}{\sqrt{\sigma\sqrt{\pi}}}e^{-\frac{t^2}{2\sigma^2}}, \qquad \psi_1(t,\sigma)=\sqrt{\frac{2}{\sigma\sqrt{\pi}}}\frac{t}{\sigma}e^{-\frac{t^2}{2\sigma^2}},$$
$$\psi_p(t,\sigma)=\frac{t}{\sigma}\sqrt{\frac{2}{p}}\,\psi_{p-1}(t,\sigma)-\sqrt{\frac{p-1}{p}}\,\psi_{p-2}(t,\sigma). \tag{2}$$

The calculation of the function in a recursive manner has some important practical advantages, as emphasized in [2] and [16]. The Hermite expansion, whose discrete counterpart is referenced as Hermite transform, assumes that the analyzed signal can be represented as the summation of the weighted basis functions [1]-[16]:

$$f(t)=\sum_{p=0}^{M-1}c_p\psi_p(t,\sigma) \tag{3}$$

where c_p denotes the p-th order Hermite coefficient, defined via the following integral:

$$c_p=\int_{-\infty}^{\infty}f(t)\psi_p(t,\sigma)dt=\int_{-T_\sigma}^{T_\sigma}f(t)\psi_p(t,\sigma)dt, \tag{4}$$

with $p=0,...,M-1$, where the compact time-support of the signal is assumed, i.e. $f(t)=0$, $t\notin[-T_\sigma,T_\sigma]$. The number of basis functions is denoted as M. For the numerical calculation of the integral (4) quadrature approximation techniques are engaged. Since it provides significant calculation advantages over other approximations, the Gauss-Hermite quadrature, defined by:

$$c_p=\frac{1}{M}\sum_{m=1}^{M}\frac{\psi_p(t_m,\sigma)}{[\psi_{M-1}(t_m,\sigma)]^2}f(t_m),\ p=0,1,...,M-1 \tag{5}$$

is commonly used, where t_m denote zeros of the M-th order Hermite polynomial, which satisfy $t_1<t_2<...<t_M$.

An infinite number $M\to\infty$ of Hermite functions is needed for the exact representation of the continuous signal $f(t)$. However, in numerous applications, a finite number of $M<\infty$ Hermite functions can be used for the signal representation with a certain approximation error [2], [16]. In contrary to continuous signals, for the case of discrete signals, Hermite transform is an orthonormal and complete signal representation if certain conditions are met.

Namely, in that case any discrete signal of length M, sampled at points t_m proportional to the roots of the M-th order Hermite polynomial, the Gauss-Hermite quadrature formula (5) provides the exact value of the integral (4), and the analyzed signal can be uniquely represented in the domain of the discrete Hermite transform.

The time axis scaling factor σ, used to "stretch" and "compress" Hermite functions relatively to the analyzed signal $f(t)$. As it is proposed in [2] and [4] and we can fix σ = 1 and introduce an equivalent parameter λ to "stretch" and "compress" the signal $f(t)$ relatively to the Hermite function basis.

For the sake of simplicity, the scaling factor σ = 1 will be omitted from the basis functions notation, and the signal scaling factor λ will be used instead in the signal notation.

In order to emphasize the difference between mentioned uniform and non-uniform sampling approaches, they are illustrated in Fig. 2, for the signal of length M = 51.

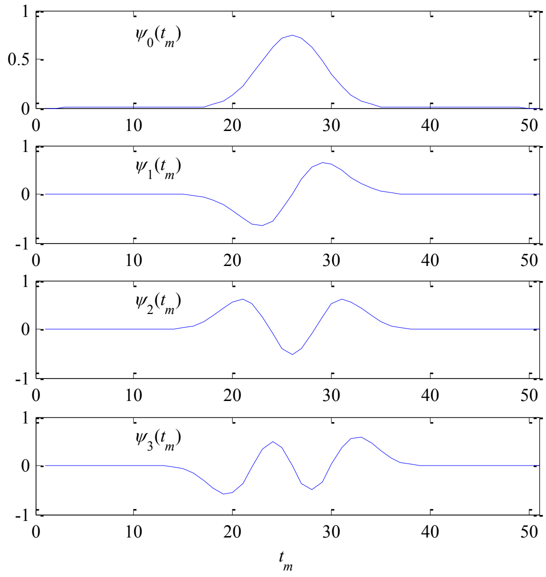

Figure 1. First four Hermite functions, where M = 51, and $\sigma = 1$

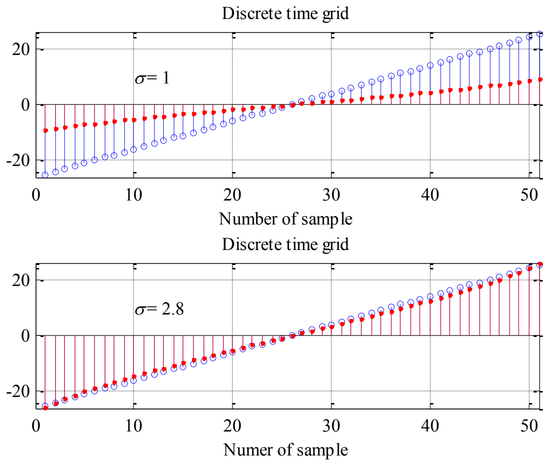

Figure 2. Discrete time grid according to the sampling theorem (blue circles) and time points proportional to the roots of the Hermite polynomial (red dots). Two values of σ are used: 1 and 2.8.

Namely, discrete time grids in two cases are shown: the grid obtained by uniform sampling and the one corresponding to the points equal to zeros of the M = 51-st Hermite polynomial are shown in upper figure, while the uniform grid and the one proportional to the roots of the Hermite polynomial by a factor $\sigma = 2.8$ is shown in the lower figure. It is important to notice that if a proper scaling factor is used, non-uniform sampling points can be mainly placed in time intervals between two neighboring sampling points.

To summarize the described facts, the inverse (3) and direct Hermite transform (5), can be written in matrix-vector notation. Let us introduce the Hermite transform matrix as:

$$\mathbf{W}_H = \frac{1}{M}\begin{bmatrix} \frac{\psi_0(t_1)}{\psi_{M-1}^2(t_1)} & \frac{\psi_0(t_2)}{\psi_{M-1}^2(t_2)} & \cdots & \frac{\psi_0(t_M)}{\psi_{M-1}^2(t_M)} \\ \frac{\psi_1(t_1)}{\psi_{M-1}^2(t_1)} & \frac{\psi_1(t_2)}{\psi_{M-1}^2(t_2)} & \cdots & \frac{\psi_1(t_M)}{\psi_{M-1}^2(t_M)} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\psi_{M-1}(t_1)}{\psi_{M-1}^2(t_1)} & \frac{\psi_{M-1}(t_2)}{\psi_{M-1}^2(t_2)} & \cdots & \frac{\psi_{M-1}(t_M)}{\psi_{M-1}^2(t_M)} \end{bmatrix}_{M\times M}. \tag{6}$$

If we introduce the vector $\mathbf{c} = [c_0, c_1, \ldots, c_{M-1}]^T$ consisted of Hermite coefficients c_p, p = 1, 2,…, M - 1 and vector $\hat{\mathbf{f}} = \left[f(\lambda t_1), f(\lambda t_2), \ldots, f(\lambda t_M) \right]^T$ consisted of M signal samples, sampled at points proportional to the roots of the M-th order Hermite polynomial, $\lambda t_1, \lambda t_2, \ldots, \lambda t_M$ the summation (5) can be written as:

$$\mathbf{c} = \mathbf{W}_H \hat{\mathbf{f}}. \tag{7}$$

Having in mind the form of the expansion (3), the inverse transform matrix is consisted of M Hermite functions:

$$\mathbf{\Psi} = \begin{bmatrix} \psi_0(t_1) & \psi_1(t_1) & \cdots & \psi_{M-1}(t_1) \\ \psi_0(t_2) & \psi_1(t_2) & \cdots & \psi_{M-1}(t_2) \\ \vdots & \vdots & \ddots & \vdots \\ \psi_0(t_M) & \psi_1(t_M) & \cdots & \psi_{M-1}(t_M) \end{bmatrix}_{M\times M} = \mathbf{W}_H^{-1}. \tag{8}$$

Based on the previous matrix definitions, the Hermite transform for the case of discrete signals reads:

$$\hat{\mathbf{f}} = \mathbf{W}_H^{-1}\mathbf{c} = \mathbf{\Psi}\mathbf{c}. \tag{9}$$

III. SAMPLING AT NON-UNIFORM POINTS

Let us consider a continuous-time signal $f(t)$ with a compact time support, such that $f(t) \neq 0$ for $-T_\sigma \le t \le T_\sigma$, sampled uniformly to obtain the corresponding finite duration discrete-time signal $f(n)$, of length $M = 2K+1$, $n = -K, \ldots, K$ according to the sampling theorem, with Δt being the sampling period. The continuous-time signal can be reconstructed, according to the sampling theorem, by using the following relation:

$$f(t) = \sum_{n=-K}^{K} f(n\Delta t)\frac{\sin\left(\pi(t-n\Delta t)/\Delta t\right)}{\pi(t-n\Delta t)/\Delta t}. \tag{10}$$

If one is interested to resample the signal at points proportional to the roots of the M-th order Hermite polynomial, $\lambda t_1, \lambda t_2, \ldots, \lambda t_M$ according to (10) we further have:

$$f(\lambda t_m) = \sum_{n=-K}^{K} f(n\Delta t)\frac{\sin\left(\pi(\lambda t_m-n\Delta t)/\Delta t\right)}{\pi(\lambda t_m-n\Delta t)/\Delta t}, \tag{11}$$

where $m = 1, \ldots, M$.

The uniformly sampled signal and the corresponding Hermite transform now can be related by combining (7) and (11) as:

$$\begin{aligned} c_p &= \frac{1}{M}\sum_{m=1}^{M}\frac{\psi_p(t_m,\sigma)}{\left[\psi_{M-1}(t_m,\sigma)\right]^2} f(\lambda t_m) = \\ &= \sum_{m=1}^{M}\sum_{n=-K}^{K}\frac{\psi_p(t_m,\sigma)}{\left[\psi_{M-1}(t_m,\sigma)\right]^2}\frac{\sin\left(\pi(\lambda t_m-n\Delta t)/\Delta t\right)}{\pi(\lambda t_m-n\Delta t)/\Delta t}\frac{f(n\Delta t)}{M} \end{aligned} \tag{12}$$

where p = 0, 2, …, M -1, m = 1,…, M. In other words, (12) is used to calculate Hermite transform coefficients of a uniformly sampled signal.

IV. REPRESENTATION OF QRS COMPLEXES

QRS complexes are important parts of ECG signals, and play an important part in medical diagnosis and treatment [1]-[5]. Due to the visual similarity of QRS complexes and Hermite functions, Hermite transform has been widely employed for the representation of these signals, with different aims. Namely, since these signals can be represented with as small number of large Hermite coefficients, Hermite transform has been employed in compression algorithms developed particularly for QRS complexes [2], [4].

In order to confirm previous theoretical considerations, and having in mind that the Hermite basis is suitable for their representation, we analyze uniformly sampled ECG signals obtained from the online MIT-BIH ECG Compression Test Database [18]. In this database, every ECG signal is recorded with two channels, known as leads. ECG signal with two leads from this database, is shown on Fig. 3. First 10 seconds of the signal are shown. The signals are available in discrete form, sampled according to the Shannon-Nyquist theorem, with sampling period $\Delta t = 1/250$ [s]. The database is consisted of 168 ECG signals and 1486 detectable QRS complexes.

We have further isolated one QRS complex, and employed (12) for its representation. Two different scaling factors λ were tested to scale the time axis of the signals, in the calculation of (12). Results are shown in Fig. 4.

It can be further observed that the scaling factor λ has an important role in the representation of discrete signals with compact support. Namely, if the basic aim is to represent the signal with a small number of coefficients, then an optimal choice of the scaling factor must be made. The value of the scaling factor depends on the application of the transform, and it is the topic of our current research [19].

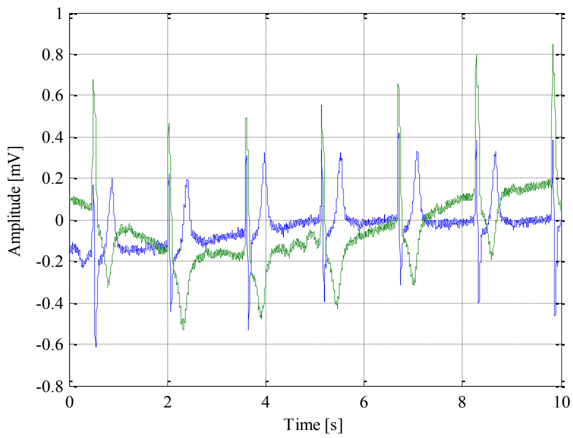

Figure 3. An example of ECG signal with two leads.

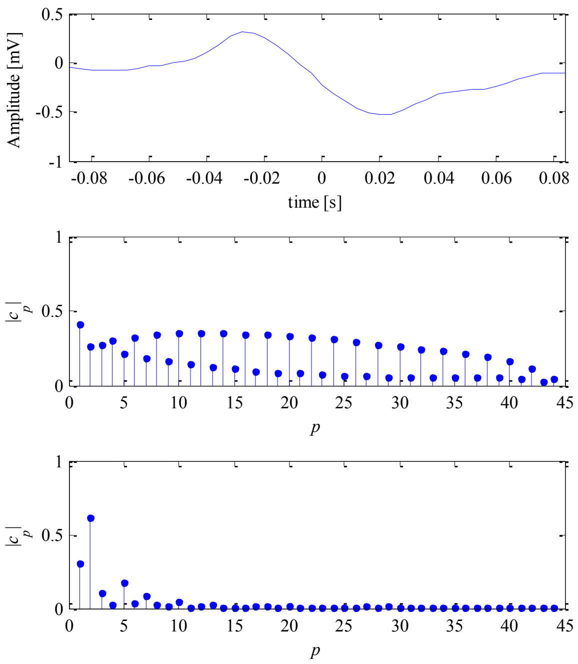

Figure 4. Hermite transform of the QRS complex: first row – the selected QRS complex, second row – Hermite coefficients of the signal with $\lambda = 1 \cdot \Delta t$, third row – Hermite coefficients of the signal with $\lambda = 5.7 \cdot \Delta t$.

V. CONCLUSION

The Hermite transform of discrete-time signals sampled according to the sampling theorem is revisited in the paper. Representation of such signals is of a great interest in practice, since the signals are mainly available in their sampled form. The analyzed signals are resampled at the points proportional to the roots of the Hermite polynomial, by reconstructing their exact values at the points of the interest, engaging the theory of the sampling theorem.

The influence of the time-scaling factor is emphasized, and the presented theory is illustrated by a numerical example with real signal. Namely, we have applied the Hermite transform on a uniformly sampled QRS complex, by resampling it at the points proportional to the roots of the Hermite polynomial. We have concluded that the scaling factor is very important for a concise representation of signals with compact time support.

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