

# The Optimization of the Hermite transform: Application Perspectives and 2D Generalization

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**Abstract** — This paper studies the discrete Hermite transform applicability in concise representation of short-term and windowed sinusoidal signals. Namely, the Hermite functions show similar behavior with windowed multi-tone signals and filtered tones, which opens the possibility for signal sparsification using an optimal transform scaling factor. In other words, the scaling factor is optimized in order to enhance the transform coefficients concentration. The scaling factor optimization method is based on concentration measures and it is further generalized to the case of 2D Hermite transform. Numerical examples illustrate the presented theoretical framework.

**Keywords** — Digital signal processing, Hermite transform, scaling factor optimization, tone signals.

## I. INTRODUCTION

THE Hermite transform has been widely studied as an alternative to the commonly used Fourier transform, since it provides a concise representation of many signals arising in different applications [1]-[10]. Namely, the ultra-wideband (UWB) communication signals and ECG signals are the most representative examples [2], [3], [10]. An intensive research has been conducted recently towards the possibilities of concise representation of QRS complexes, the most characteristic waves of ECG signals, having a significant role in medical diagnosis and treatment [1]-[4],[8]-[10]. The main motivation for these approaches comes from the visual similarity of UWB Gaussian doublets and QRS complexes with the basis functions of Hermite transform. Hence, powerful compression algorithms for QRS complexes have been developed, exploiting the use of the Hermite transform as a crucial step for compression [2],[3]. Moreover, the classification and detection of QRS complexes have been also widely studied in terms of the advantages provided by the Hermite representation. A compact representation of QRS complexes concentrated in a few coefficients provides a potential for efficient medical diagnosis, detection of anomalies, and hearth diseases such as arrhythmia [2], [4]. Besides afore mentioned applications, the Hermite transform is also exploited in many other research areas, including: digital image segmentation [5], computed

tomography [6], analysis of protein structure in biology, physical optics [7], radar signal processing, [9] and so on.

Exhibiting many interesting mathematical properties, which consequently spread the spectrum of possible applications, this particular signal transform was recently studied in the sparse signal processing and compressed sensing context [8], [10]. The reconstruction of signals being sparse in the Hermite domain was studied in [8]. Therein, an efficient gradient-based reconstruction approach was presented.

The parameter optimization of the Hermite transform, leading to the improvement of the signal's representation in this domain was proposed in [10]. This work emphasized the importance of the transform's concentration, and to this aim a parameterization method based on concentration measure minimization was proposed. Namely, the time-axis scaling factor and the time-shift of the basis functions were considered.

In this paper, we study the application perspectives of this optimization approach, particularly in achieving the compact representation of single-tone and multi-tone windowed signals. Moreover, as the filtering is a common processing technique applied on tone signals, we also consider a possibility to sparsify the representation of signals filtered using the common Butterworth digital filter. Next, we present the generalization of the approach to a two-dimensional Hermite transform. Namely, as the 2D Hermite transform can be calculated applying the 1D transform on both signal dimensions successively, this property is used in the development of the generalized approach.

The paper is organized as follows. A short overview of the 1D Hermite transform is presented in Section 2. Sections 3 and 4 describe the scaling factor optimization approach, and its 2D generalization, respectively. Section 4 presents the numerical results, while the paper ends with concluding remarks.

## II. THE ONE-DIMENSIONAL HERMITE TRANSFORM

Starting from the definition of the  $p$ -th order Hermite polynomial [1]-[3], [8]-[10]:

$$H_p(t) = (-1)^p e^{t^2} \frac{d^p (e^{-t^2})}{dt^p}, \quad (1)$$

the Hermite functions of the corresponding order are defined as:

$$\psi_p(t, \sigma) = \left( \sigma 2^p p! \sqrt{\pi} \right)^{-1/2} e^{-t^2/2} H_p(t/\sigma). \quad (2)$$

This work is supported by the Montenegrin Ministry of Science, project grant: "New ICT Compressive sensing based trends applied to: multimedia, biomedicine and communications (CS-ICT)" (Montenegro Ministry of Science, Grant No. 01-1002).

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The scaling factor  $\sigma$  controls the width of basis functions, and its properly chosen value may lead to a more concentrated representation. The 1D Hermite expansion of the signal  $f(t_m)$  has the following form:

$$f(t_m) = \sum_{p=0}^{M-1} c_p \psi_p(t_m, \sigma) \quad (3)$$

where  $\psi_p(t_m)$  is the  $p$ -th order Hermite basis function and  $M$  is the number of basis functions used in the expansion. Argument  $t_m$  represent sampling points equal to the roots of the  $M$ -th order Hermite polynomial. In general, an infinite number of basis functions is needed for the representation of a continuous-time signal using (3). However, if both the signal and the basis functions are sampled at points  $t_m$ , relation (3) is the accurate signal expansion, known also as the inverse discrete Hermite transform. If the Gauss-Hermite quadrature is applied in the calculation of the expansion coefficients  $c_p$ , one may define the 1D discrete Hermite transform as follows:

$$c_p = \frac{1}{M} \sum_{m=1}^M \frac{\psi_p(t_m, \sigma)}{[\psi_{M-1}(t_m, \sigma)]^2} f(t_m), p = 0, 1, \dots, M-1. \quad (4)$$

### III. NON-UNIFORM SAMPLING AND PARAMETER OPTIMIZATION

It is crucial to emphasize that the signal needs to be sampled at points  $t_m$  proportional to the roots of the  $M$ -th order Hermite polynomial. Signals are usually sampled in accordance to the sampling theorem, with samples available at uniform points, differing from the points of interest  $t_m$ . The values of the signal at the required time points  $t_m$  can be acquired by using a windowed *sinc* interpolation. Namely, assuming that the signal was sampled uniformly to obtain the corresponding finite duration discrete-time values  $f(n)$ , having an odd-length  $M = 2K+1$ ,  $n = -K, \dots, K$ , then, according to the sampling theorem, the continuous-time signal can be reconstructed and immediately resampled at the desired points  $\lambda t_1, \lambda t_2, \dots, \lambda t_M$  by applying the following interpolation [2], [10]:

$$f(\lambda t_m) \approx \sum_{n=-K}^K f(n\Delta t) \frac{\sin(\pi(\lambda t_m - n\Delta t) / \Delta t)}{\pi(\lambda t_m - n\Delta t) / \Delta t}, \quad (5)$$

with  $m = 1, \dots, M$ ,  $n = -K, \dots, K$ , and  $\Delta t$  denoting the sampling period.

Furthermore, as it is done in [2] and [10], instead of stretching and compressing basis functions varying  $\sigma$ , one may assume that  $\sigma=1$ , and stretch and compress the signal in order to match the set of basis functions. In other words, the alternative parameter  $\lambda$  can be introduced in (4) in order to represent the stretched and compressed form of the analyzed signal  $f(\lambda t_m)$ .

There is a particularly high interest in many applications to represent the signals with the smallest number of transform coefficients. To this aim, we can optimize the scaling factor  $\lambda$  such that it minimizes the concentration measure:

$$\mathcal{M} = \sum_{p=0}^{M-1} |c_p|, \quad (6)$$

that is, the optimal scaling factor can be found solving:

$$\lambda = \arg \min_{\lambda} \left\| \sum_{m=1}^M \frac{\psi_p(t_m, \sigma)}{[\psi_{M-1}(t_m, \sigma)]^2} f(\lambda t_m) \right\|_1, \quad (7)$$

with  $f(\lambda t_m)$  calculated according to (5). The problem can be solved using the following iterative algorithm [10]:

#### Algorithm 1: Scaling factor optimization

**Require:**

- Uniformly sampled signal  $f(n)$  of length  $M = 2K + 1$
- Step parameter  $\mu$

1: Set  $\lambda^{(0)} \leftarrow M\Delta t / \left[ 2 \left( \sqrt{\pi(M-1)/1.7} + 1.8 \right) \right]$

2: Set  $\Delta \leftarrow 2 / t_M$

3: Set  $\varepsilon \leftarrow 10^{-10}$

**While**  $\Delta > \varepsilon$

4:

$$\mathbf{K}_{\lambda}^+ \leftarrow \begin{bmatrix} k_{11}^+ & k_{12}^+ & \dots & k_{1M}^+ \\ k_{12}^+ & k_{22}^+ & \dots & k_{2M}^+ \\ \vdots & \vdots & \ddots & \vdots \\ k_{M1}^+ & k_{M2}^+ & \dots & k_{MM}^+ \end{bmatrix}, \mathbf{K}_{\lambda}^- \leftarrow \begin{bmatrix} k_{11}^- & k_{12}^- & \dots & k_{1M}^- \\ k_{12}^- & k_{22}^- & \dots & k_{2M}^- \\ \vdots & \vdots & \ddots & \vdots \\ k_{M1}^- & k_{M2}^- & \dots & k_{MM}^- \end{bmatrix}$$

$$k_{ij}^{\pm} = \frac{\sin[\pi((\lambda \pm \Delta)t_i - (j - K - 1)\Delta t) / \Delta t]}{\pi((\lambda \pm \Delta)t_i - (j - K - 1)\Delta t) / \Delta t}, i, j \in \{1, \dots, M\}$$

5:  $\nabla^{(k)} \leftarrow \frac{1}{M} \left( \sum_{p=0}^{M-1} |\mathcal{H}(\mathbf{K}_{\lambda}^+ \mathbf{f})| - \sum_{p=0}^{M-1} |\mathcal{H}(\mathbf{K}_{\lambda}^- \mathbf{f})| \right)$

6:  $\lambda^{(k+1)} \leftarrow \lambda^{(k)} - \mu \nabla^{(k)}$

7:  $\beta \leftarrow \text{sign}(\nabla^{(k)} \nabla^{(k-1)})$

8: **If**  $\beta < 0$  **then**  $\Delta \leftarrow \Delta / 2$

**End while**

9: **Return**  $\lambda^{(k)}$

In the previous algorithm, the vector  $\mathbf{f}$  contains uniformly sampled signal values, and matrices  $\mathbf{K}_{\lambda}^{\pm}$  are used for the calculation of (5). The operator  $\mathcal{H}(\cdot)$  denotes the Hermite transform (4).

### IV. THE 2D TRANSFORM AND ITS OPTIMIZATION

The proposed scaling factor algorithm can be generalized to the 2D case. As for the 1D case, the signals that have similar waveform as some of Hermite basis functions are suitable for representation, in the sense of having a possibly compact support and a potential for compression in the Hermite transform domain. Let us observe the 2D signal  $f(t_m, t_n)$  of size  $M \times N$ . The inverse discrete 2D Hermite transform of this signal is defined as follows:

$$f(t_m, t_n) = \sum_{p=0}^{M-1} \sum_{k=0}^{N-1} c_{pk} \psi_p(t_m, \sigma_1) \psi_k(t_n, \sigma_2) \quad (8)$$

while the 2D discrete Hermite transform is defined by:

$$c_{pk} = \frac{1}{MN} \sum_{m=1}^M \sum_{n=1}^N \frac{\psi_p(t_m)}{[\psi_{M-1}(t_m)]^2} \frac{\psi_k(t_n)}{[\psi_{N-1}(t_n)]^2} f(\lambda_1 t_m, \lambda_2 t_n) \quad (9)$$

with  $t_m$  and  $t_n$  being the roots of the  $M$ -th order and  $N$ -th order Hermite polynomials, and with the similar introduction of signal time-axis scaling factors  $\lambda_1$  and  $\lambda_2$  as in 1D case, with fixed  $\sigma_1 = 1, \sigma_2 = 1$ . It is well-known that the 2D Hermite functions can be calculated as a product of the corresponding 1D Hermite functions [11]:

$$\psi_{pk}(t_m, t_n) = \psi_p(t_m) \psi_k(t_n). \quad (10)$$

If we observe (9) and (8), it can be easily concluded that both the direct and the inverse Hermite transform can be calculated by using the 1D transform pair (4) and (3) over one variable with the other fixed, and then repeating the same calculation procedure for the second variable. Hence, in order to find scaling factors  $\lambda_1$  and  $\lambda_2$  producing the most concentrated transform, we solve:

$$(\lambda_1, \lambda_2) = \arg \min_{\lambda_1, \lambda_2} \sum_{p=0}^{M-1} \sum_{k=0}^{N-1} c_{pk}, \quad (11)$$

by applying the Algorithm 1 in both dimensions successively. The scaling factors, producing the best possible concentrations over the observed dimension, are chosen.

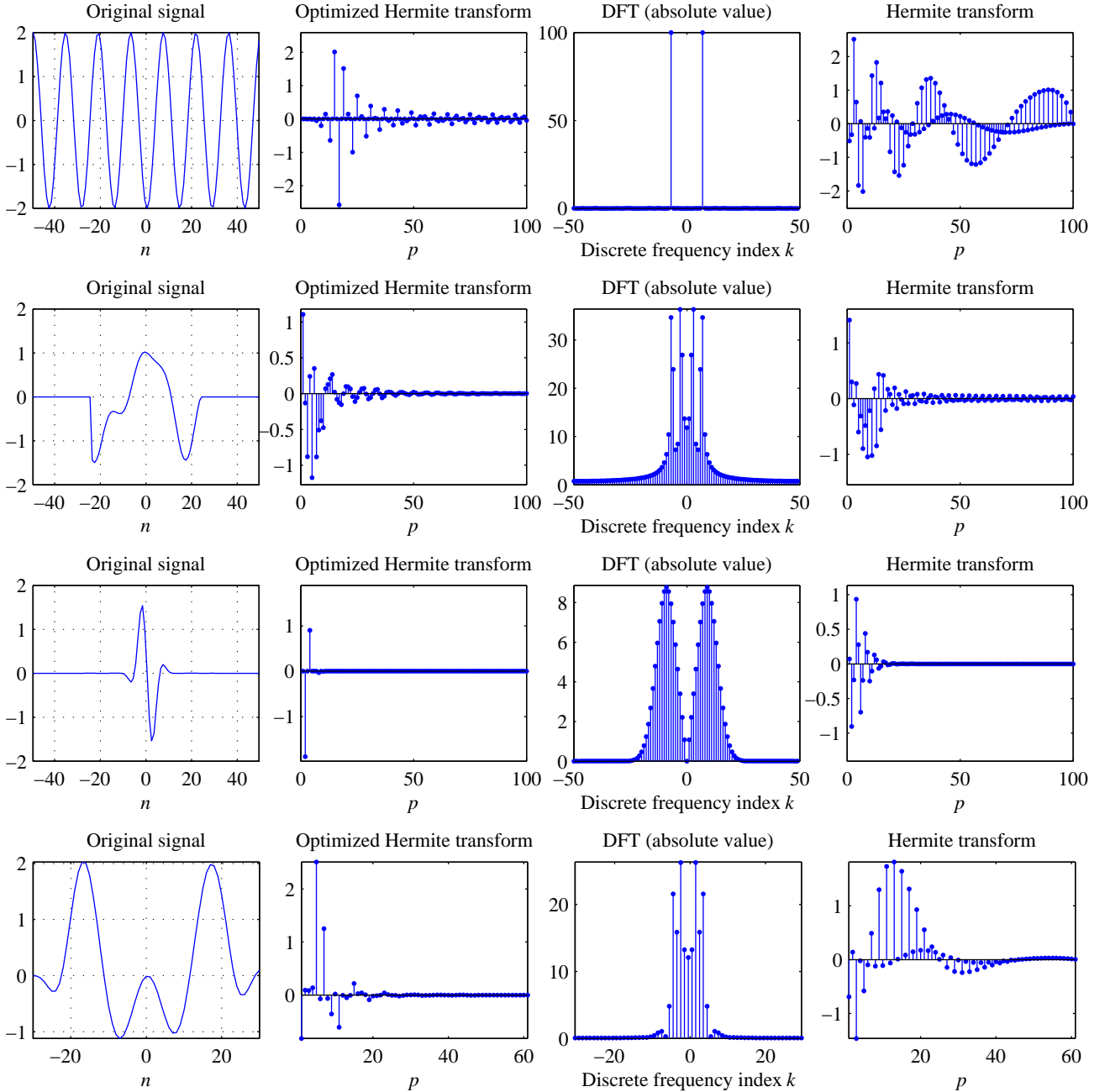


Fig. 1. The Hermite transform optimization results for different types of tone signals: first row - Pure sinusoid, second row - Short-term multi-tone sinusoidal signal, third row: Windowed multi-tone sinusoidal signal (Gaussian window), fourth row - Filtered multi-tone sinusoidal signal

## V. NUMERICAL RESULTS AND DISCUSSION

In order to illustrate and clarify the applicability perspective of the approach we, observe some characteristic cases.

*Example 1:* Let us observe the case of a pure sinusoid of the form (pure tone):

$$f(n) = -2 \cos(14\pi n / M), \quad (12)$$

of length  $M = 100$ ,  $-M/2 \leq n \leq M/2 - 1$ . The results presented in Fig. 1 (first row) illustrate the fact that the Fourier transform is an optimal representation for this type of signals, outperforming the Hermite domain approach in sense of the coefficient's concentration.

*Example 2:* Let us observe the case of short-duration sinusoids appearing, for example, in the communications:

$$f(n) = \begin{cases} \sum_{i=1}^3 A_i \sin\left(\frac{\omega_i n}{M} + \phi_i\right), & -M + 25 \leq n \leq M - 25 \\ 0, & -M \leq n \leq -M + 26 \text{ and } M - 26 \leq n \leq M \end{cases} \quad (13)$$

where  $M = 100$ , with amplitudes taking values  $A_i \in \{-1, 1\}$ , frequencies from the set  $\omega_i \in \{13.5\pi, 5.267\pi, 12.1\pi\}$  and phases taking values  $\phi_i \in \{0, \pi/2, 0\}$  for  $i = 1, 2, 3$  respectively. Analyzing the results in Fig. 1 (second row), we may conclude that the Hermite transform (optimized using the presented approach) outperforms the Fourier transform in this case and provides a more compact (compressible) representation in the transform domain.

*Example 3:* Consider now the windowed form of the sinusoidal signals (Gaussian window is applied):

$$f(n) = -2 \sin(18\pi n / M) \exp(-(n/2\sigma_0)^2), \quad (14)$$

with  $\sigma_0 = 3.5$ ,  $-M/2 \leq n \leq M/2 - 1$ . Obviously, the considered optimized Hermite transform is the most optimal transform for windowed sinusoids, unlike the Fourier transform which is spread over half of the frequency band Fig. 1 (third row).

*Example 4:* An interesting application scenario could include a filtered multi-tone signal of the form:

$$f(n) = \cos(5.8\pi n / M) - \cos(3\pi n / M) \quad (15)$$

with  $M = 50$ , defined at discrete instants  $-2M/2 \leq n \leq M/2$  (using low-pass Butterworth filter of 5<sup>th</sup> order, and normalized cut-off frequency 0.2 (1 corresponds to the half the sample rate)). It can be observed that the Hermite transform (optimized using the proposed approach) outperforms the Fourier transform based representation in this example as well, as shown in Fig. 1 (last row).

*Example 5:* Let us observe a 2D localized signal that has a suitable representation in 2D HT due to the visual similarity with basis functions:

$$f(m, n) = -3 \sin\left(\frac{5\pi(m+n)}{M^2}\right) \exp\left(-\frac{m^2}{2\sigma_0^2} - \frac{n^2}{2\sigma_1^2}\right) \quad (16)$$

defined for instants  $-15 \leq m \leq 15$  and  $-15 \leq n \leq 15$  where  $M = 31$ ,  $\sigma_0 = \sigma_1 = 1.25$ . Results are shown in Fig. 2, confirming the fact that the optimal concentration of the signal can be achieved in the 2D signal case.

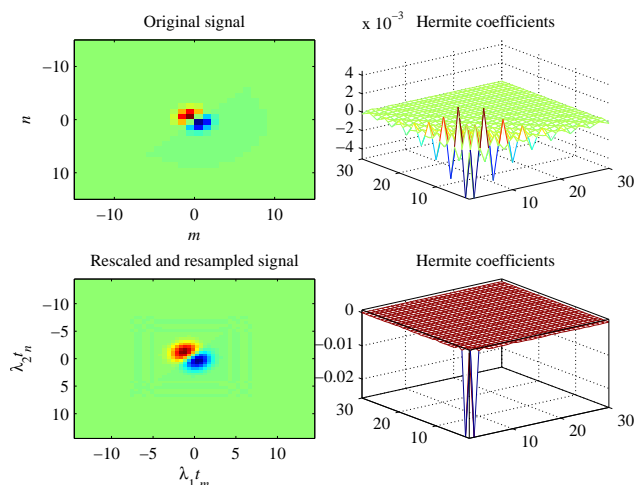


Fig. 2. The illustration of the 2D Hermite transform optimization in the representation of a 2D windowed sinusoid.

## VI. CONCLUSION

In this paper we studied the application perspectives of the Hermite transform scaling factor optimization approach. It has been shown that the optimization approach provides improved Hermite representation of windowed single- and multi-tone signals. Moreover, the optimization approach is generalized by extending the concept to 2D signals.

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