Sparse Signal Reconstruction Based on Random Search Procedure

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***Abstract*—A method for reconstruction of sparse signals is presented in this paper. It is an improved version of the direct-search method for finding the set of non-zero coefficients representing the solution in sparsity domain. The proposed random search procedure is performed assuming the largest possible number of non-zero coefficients still satisfying the available measurements system. In the sparse signal processing and compressive sensing theory, this number should be smaller than or equal to the number of measurements. For each possible arrangement of the examined non-zero coefficients, the reconstruction is done by solving the system of equations in the least square sense, until the solution is found. Benefits of the proposed method are discussed. The calculation complexity improvement of the proposed method, compared to the direct-search, is analytically expressed. It depends on the total number of signal samples, number of measurements and the signal sparsity. The presented theory is confirmed with numerical examples.**

***Keywords* — compressive sensing, signal reconstruction, direct search, sparsity**

I. INTRODUCTION

A signal with a small number of nonzero coefficients (sparse signal) can be reconstructed from a reduced set of available samples/measurements [1]–[11]. The reduced number of measurements can be a consequence of various circumstances. It can occur as a result of a sampling strategy developed with the aim to reduce the storage requirements for the data. Measurements can be unavailable due to physical constraints or their intentional omitting due to a high noise corruption [11]. The analysis and reconstruction of sparse signals was the topic of many research papers [1]– [15]. Numerous reconstruction theorems and algorithms were developed [1], [2], [4]–[8].

Many signals in real applications can be considered as sparse in a certain transformation domain, meaning that the idea of sparse signal reconstruction can be exploited in different areas of signal processing. For example, digital images can be considered as sparse in the domain of discrete cosine transform (DCT), whereas radar ISAR data are sparse in the domain of two-dimensional Fourier transform [16].

One of the challenging topics in the compressive sensing is the optimal sampling strategy that will allow to reconstruct the signal with smallest possible number of available samples [12]. Various approaches are used to this aim, like those that minimize the coherence index of the isometry constant for a given signal transform. The direct search method provides exact reconstruction results if the unique reconstruction is theoretically possible. However, this method is computationally complex. For a large dimension of the reconstruction problem, number of available samples and sparsities, it is NP-hard and therefore not computationally feasible.

In this paper, we introduce an improvement in the computation of signal reconstruction by modifying the direct search strategy. It is known that for an observed sparsity level, compressive sensing algorithms, for instance ℓ_1-norm based or iterative, greedy, and other proposed algorithms [5], require a larger number of available measurements than the one required in ℓ_0-norm minimization. The aim is to overcome this issue by exploiting the possibility to reduce the number of trials in the direct search reconstruction procedure.

The paper is organized as follows. Basic compressive sensing definitions are presented in Section II. In Section III the direct search reconstruction algorithm is presented. The proposed, random search method, is introduced in section Section IV. Results and comparison are shown in Section V whereas the paper ends with concluding remarks.

II. BASIC DEFINITIONS

Let us consider a complex-valued discrete signal $x(n)$ of length N and its corresponding transformation domain $X(k)$

$$x(n)=\sum_{k=0}^{N-1}X(k)\psi_k(n), \qquad X(k)=\sum_{n=0}^{N-1}x(n)\varphi_n(k), \quad (1)$$

or in vector form $\mathbf{x}=\mathbf{\Psi}\mathbf{X}$ and $\mathbf{X}=\mathbf{\Phi}\mathbf{x}$. The inverse and the direct transform matrices are denoted as $\mathbf{\Psi}$ and $\mathbf{\Phi}$, respectively. We say that a signal is K-sparse in the transformation domain if the number of nonzero coefficients K is much smaller than the total length of signal N, i.e., $K \ll N$. Then a sparse signal can be reconstructed with $M < N$ measurements. The signal with M samples/measurements available is denoted as $y(m)$

$$y(m)=\sum_{k=0}^{N-1}X(k)\psi_k(m). \quad (2)$$

Previous definition can be written in a vector form as

$$\mathbf{y}=\mathbf{A}\mathbf{X} \quad (3)$$

This work is supported by the Montenegrin Ministry of Science, project grant funded by the World Bank loan: CS-ICT "New ICT Compressive sensing based trends applied to: multimedia, biomedicine and communications".

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where $\mathbf{A}$ is the measurement matrix of size $M \times N$. It is formed based on the matrix $\mathbf{\Psi}$, containing rows that correspond to the positions of the available measurements/observations, whereas the rows corresponding to the missing samples are omitted.

The sparse signal reconstruction can be defined as the solution of the optimization problem

$$\min \ \|\mathbf{X}\|_0 \ \text{ subject to } \ \mathbf{y} = \mathbf{A}\mathbf{X}. \tag{4}$$

Having an undetermined system of linear equations defining the available measurements, the solution of the signal reconstruction problem is the one satisfying this system of equations, and being the sparsest possible. That is, the aim is to minimize the sparsity of $\mathbf{X}$ using the available measurements $\mathbf{y}$. This is achieved by exploiting a sparsity measure. A natural choice for this measure is the so-called ℓ_0-norm which counts the number of nonzero coefficients in $\mathbf{X}$, although not satisfying norm properties in a strict mathematical sense. However, this function is not convex and its minimization could be done only through a combinatorial search. Moreover, it can be easily shown that a direct combinatorial search is not computationally feasible for a reasonable length of the considered signal, its sparsity and number of available samples. This pseudo-norm is also very sensitive to the noise influence and quantization errors. This is the reason why, in practice and theory, more robust norms are exploited as sparsity measures.

The ℓ_1-norm is the most frequent used norm since it is closest convex function to the ℓ_0-norm. It is equal to the sum of absolute values of $\mathbf{X}$. However, all norm-one reconstruction methods, as well as other standard algorithms developed within the fields of sparse signal recovery and compressed sensing, require more samples/measurements than the minimal possible number that can provide a unique signal reconstruction in theory. Motivated by this fact, we try to reduce the computational cost of the combinatorial approach, with the aim to obtain the results similar to the direct search in sense of the minimal required number of measurements needed for a successful unique reconstruction.

III. Direct search reconstruction

Any problem described with (4) can be solved by a direct search over the whole set of possible values of nonzero coefficient positions. This procedure is defined as the direct search minimisation of the ℓ_0-norm. Assume a vector $\mathbf{X}$ with sparsity K. We try to detect indices of the nonzero values $k \in \{k_1, k_2, ..., k_K\}$ out of the set of all possible indices between 1 and N

$$k \in \mathbf{K} \subset \mathbf{N} \tag{5}$$

where $\mathbf{K} = \{k_1, k_2, ..., k_K\}$, $\mathbf{N} = \{1, 2, ..., N\}$. The vector $\mathbf{X}_K$ contains assumed K nonzero elements of $\mathbf{X}$ at the positions from set $\mathbf{K}$. The system

$$\mathbf{y} = \mathbf{A}_K\mathbf{X}_K \tag{6}$$

with $M > K$ equations is solved by minimising the least square error

$$\begin{aligned} e^2 = (\mathbf{y} - \mathbf{A}_K\mathbf{X}_K)^H(\mathbf{y} - \mathbf{A}_K\mathbf{X}_K) = \\ \|\mathbf{y}\|_2^2 - 2\mathbf{X}_K^H\mathbf{A}_K^H\mathbf{y} + \mathbf{X}_K^H\mathbf{A}_K^H\mathbf{A}_K\mathbf{X}_K. \end{aligned} \tag{7}$$

The minimum of the error is found from

$$\frac{\partial e^2}{\partial \mathbf{X}_K^H} = -2\mathbf{A}_K^H\mathbf{y} + 2\mathbf{A}_K^H\mathbf{A}_K\mathbf{X}_K = 0. \tag{8}$$

The solution is calculated as

$$\begin{aligned} \mathbf{A}_K^H\mathbf{A}_K\mathbf{X}_K &= \mathbf{A}_K^H\mathbf{y} \\ \mathbf{X}_K &= \left(\mathbf{A}_K^H\mathbf{A}_K\right)^{-1}\mathbf{A}_K^H\mathbf{y}. \end{aligned} \tag{9}$$

For all solutions we check the error $\mathbf{y} - \mathbf{A}_K\mathbf{X}_K$. The reconstruction of the signal $\mathbf{X}$ is exact when the mean square error is equal to zero. The reconstruction is not unique if there is more than one solution.

IV. Random search reconstruction

In the direct search procedure we should check all combinations of K nonzero out of N coefficients in total. To find all possible combinations of $\{k_1, k_2, ..., k_K\} \subset \mathbf{N}$, the total number of combinations is equal to

$$\binom{N}{K} \tag{10}$$

and it could be very large. The expected number of checked combinations is

$$T_d = \frac{1}{2}\binom{N}{K}. \tag{11}$$

Even though the direct search is an accurate method, it is computationally not feasible to get a solution for a large signal. For the random search procedure, we will consider a system with M unknowns. Taking $M-1 > K$ equations in a combination, less trials will be needed to find the solution. Let us consider a random combination of $M-1$ nonzero positions. The new system is then

$$\mathbf{y} = \mathbf{A}_{sel}\mathbf{X}_{sel} \tag{12}$$

with M equations. If the considered combination includes all K nonzero positions from $\mathbf{X}$, then the system (12) of M equations with $M-1$ unknowns have the unique solution. Note that only K coefficients in the solution are nonzero and remaining $M-1-K$ coefficients are zero valued.

Considering the new system (12), the probability that we find the solution is

$$P_s = \frac{\binom{N-K}{M-1-K}}{\binom{N}{M-1}}. \tag{13}$$

The expected number of trials can be estimated as

$$T_r = \frac{1}{P_e} = \frac{\binom{N}{M-1}}{\binom{N-K}{M-1-K}}. \tag{14}$$

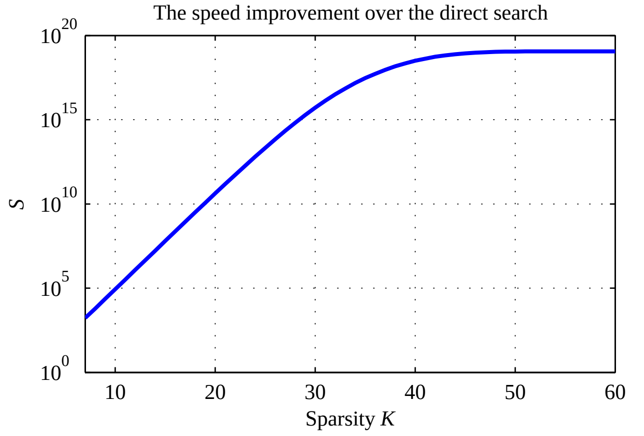

Fig. 1. The improvement in the calculation complexity over the direct search, when the random search is applied on signals with various sparsity K, length $N = 128$ and number of available samples $M = 2K + 1$.

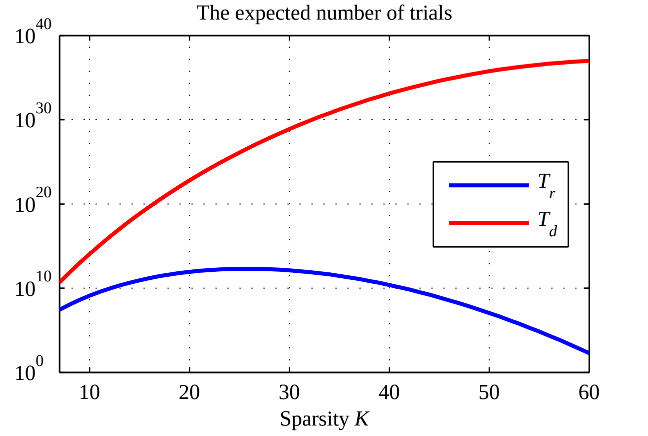

Fig. 2. The expected number of trials in direct search and random search for signal of length $N = 128$ with various sparsity K and number of available samples $M = 2K + 1$.

The improvement in speed of the proposed random search procedure compared to the direct-search is

$$S = \frac{T_d}{T_r} = \frac{1}{2}\frac{\binom{N}{K}\binom{N-K}{M-1-K}}{\binom{N}{M-1}}. \tag{15}$$

The amount of the speed improvement will be illustrated by several examples in the next section.

V. Examples

Example 1: Let us consider a signal of length $N = 128$, having $M = 15$ available samples and sparsity $K = 7$. The total number of direct-search combinations for this case is

$$\binom{N}{K} \sim 10^{10}.$$

Probability that we guess solution by proposed method is

$$P_s \approx 3.6 \times 10^{-8}$$

whereas the expected number of trials equals

$$T_r \approx 2.75 \times 10^{7}.$$

The proposed method is $S \approx 1700$ times faster than direct-search (in average).

Let us now observe a signal having the same length $N = 128$, and $M = 31$ available samples with sparsity $K = 15$. Following the previous analysis, the calculation speed-up is

$$S \approx 7.7 \times 10^{7}.$$

The expected number of trials in the random search procedure applied in this case is

$$T_r \approx 8.5 \times 10^{10}.$$

Example 2: For signal of length $N = 128$ we vary sparsity K from 7 to 60 and assume that the number of available samples is $M = 2K+1$ for each observed sparsity. The speed improvement of the proposed method over the direct search, calculated according to (15) is shown in Fig. 1. The expected trials number is also calculated for both procedures, according to (11) and (14). The results are shown in Fig. 2.

Example 3: The main motivation to introduce an improved version of the direct search lies in the fact that standard sparse reconstruction algorithms, including those based on ℓ_1-norm require a larger number of available samples for the recovery of missing samples than the reconstruction based on the corresponding direct search aiming to minimize the ℓ_0-norm. In order to illustrate this issue, we observe a $N = 20$ length signal K-sparse in the discrete Fourier transform domain. The signal has the following form

$$x(n) = \sum_{i=1}^{K} A_i e^{j2\pi k_i \frac{n}{N}}, \tag{16}$$

with amplitudes and frequencies having random values with uniform distribution, satisfying $0 \le A_i \le 2$ and $0 \le k_i \le N-1$. Number of available samples was varied from $M = 1$ to $M = 19$ whereas for each number of available samples sparsity was varied from $K = 1$ to $K = 19$. Note that in cases when $K > M$ the reconstruction is not possible.

The proposed random search is compared with Orthogonal Matching Pursuit (OMP), a representative algorithm from the compressive sensing framework introduced in [5]. The random search procedure terminates when the solution is found, or the number of trials exceeds $\binom{N}{M-1}$. The experiment was conducted based on 100 independent realizations of signals with random missing samples positions and the probability of successful reconstruction is calculated.

The results are shown in Figs. 3 and 4. Comparing these results, it can be seen that the OMP-based reconstruction requires a larger number of available samples M (for a given sparsity K) than the corresponding random search procedure. For example, in the OMP case with $K = 2$ accurate reconstruction in 100% of trials requires exactly $M = 9$ available

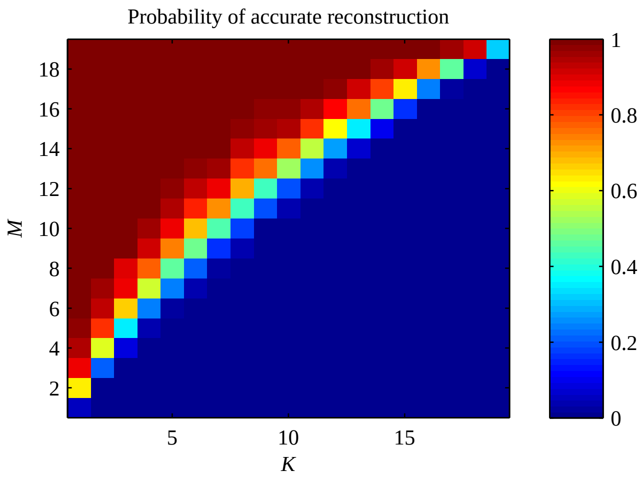

Fig. 3. The probability of successful reconstruction using OMP as an example of standard sparse signal recovery algorithms. The results, shown for various K and M are obtained based on 100 independent realizations of signals with random missing samples positions, amplitudes and frequency positions. The signal length is $N = 20$.

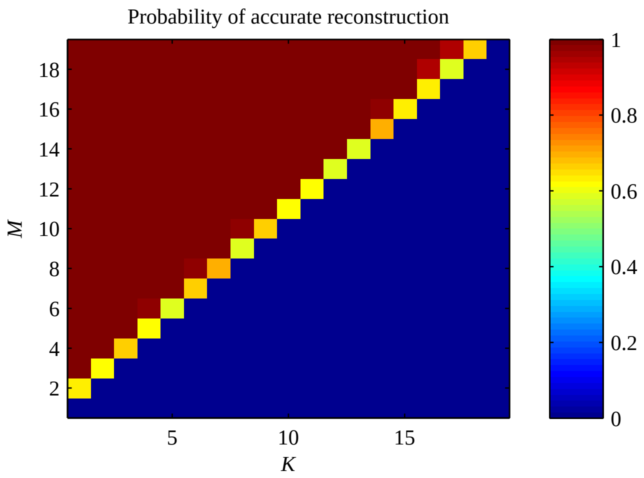

Fig. 4. The probability of successful reconstruction using the proposed random search procedure. The results, shown for various K and M are obtained based on 100 independent realizations of signals with random missing samples positions, amplitudes and frequency positions. The signal length is $N = 20$.

samples, whereas the proposed procedure requires only $M = 4$ available samples.

VI. Conclusions

In this paper we propose a method for sparse signal reconstruction based on the random search for K nonzero coefficient positions in the sparsity domain. Since we have M available measurements, in each trial $M-1$ randomly selected positions are considered. If the considered combination includes all the nonzero coefficients, then the reconstruction is done successfully. Transform coefficients with wrongly assumed nonzero values are automatically set to zero using the partial sensing matrix pseudo-inversion involved in the signal reconstruction. Taking more positions in one trial will converge to the solution faster than the direct-search procedure. The random search method is compared with the direct-search, showing noticeable improvement in the calculation cost. The basic motivation for this research is the fact that standard algorithms from the compressed sensing framework require a larger number of available samples for the successful reconstruction than it is required by the direct search based reconstruction. This issue is illustrated in comparison with OMP algorithm. The obtained results confirm that the calculation improvements in the direct search based signal recovery represent important and open topics for further research.

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