

The Reconstruction of 2D Sparse Signals By Exploiting Transform Coefficients Variances

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Abstract—In this paper, a compressed sensing approach for the reconstruction of 2D sparse signals with missing samples is presented. The statistical behavior of transform coefficients in the case of randomly undersampled signals is exploited as the basis of a simple algorithm for the signal support detection. With the detected signal support various reconstruction methods can be applied in the signal recovery: non-iterative reconstruction for signals with close transform coefficient values, matching pursuit based iterative reconstruction, or the combination of these two methods. As the case study, 2D discrete Fourier transform is observed, commonly appearing in radar imaging applications. The theory is confirmed through several numerical experiments, including the illustration of the applicability in the ISAR image reconstruction.

Keywords—Compressed sensing; digital signal processing; ISAR imaging; radar signals; sparse signal processing

I. INTRODUCTION

Compressed sensing (CS), as a recently developed theory in signal processing, attracted significant research attention during the last decade [1]-[22]. Providing different reconstruction methods for signals having missing samples (measurements, observations), various potential applications of this theory have been identified. The reconstruction of radar signals, digital images and biomedical signals with unavailable data are just some representative examples of CS applications [12]-[16], [19]. For the context of this paper, especially interesting are the applications in radars, namely the ISAR imaging [13]-[22].

Regardless the considered application, one certain condition is required for possible recovery of missing samples: it is assumed that the signals are sparse in some transformation domain [8]-[10]. The sparsity denotes that the considered signal can be represented with a small number of non-zero transform coefficients [6]. For example: digital images are known for sparsity (or approximate sparsity) in the DCT domain, QRS complexes in ECG signals are well-known for their sparsity in the domain of Hermite transform [12], whereas ISAR signals appearing in radar applications are sparse in the domain of 2D Discrete Fourier Transform (DFT) [19], [21]. On the other side, in the measurement domain (usually discrete-time domain) these signals are dense.

The signal samples unavailability may arise as a consequence of sampling strategy in order to compress the sensed data. On the other side, signal measurements can be unavailable due to the physical constraints. In certain applications, signal samples can be highly corrupted by noise, and can be intentionally omitted in further processing. All these cases can be treated in the same way by the CS theory and reconstruction algorithms [1], [3], [6].

CS-based reconstruction exploits signal sparsity in the reconstruction process. This means that an adequate sparsity measure is needed for the reconstruction of missing values. The most natural choice is simply the number of non-zero values in the N -length transform vector \mathbf{X} , sometimes known as ℓ_0 (pseudo) norm [6]-[10]:

$$\|\mathbf{X}\|_0 = \sum_{k=0}^{N-1} |X(k)|^0 \quad (1)$$

However, based on the previous definition it can be easily concluded that this sparsity measure is highly sensitive even to small noise or quantization errors. Moreover, this function is non-convex, meaning that besides direct search, linear programming approaches, gradient-base approaches etc. are not applicable in the reconstruction process. This is the reason for the application of ℓ_1 -norm as the measure of signal sparsity:

$$\|\mathbf{X}\|_1 = \sum_{k=0}^{N-1} |X(k)| \quad (2)$$

which opened the possibility to apply afore mentioned iterative reconstruction approaches [6]-[10].

Besides exploiting ℓ_1 -norm in the reconstruction process, sometimes it is possible to determine the signal support by involving some, usually transform-specific theory [1]-[3], [6], [7]. Namely, some particular research results have shown that transform coefficients of randomly undersampled signals exhibit different statistical behavior at signal and non-signal coefficients positions [2]. That was the basis for the introduction of algorithms based on mean values analysis and variance analysis of transform coefficients in initial signal transform [1], [3], [6]. When the positions and number of non-zero values are determined, further reconstruction is possible under the condition that the signals are sparse, and under assumption that the reconstruction conditions are satisfied (measurement matrices incoherence etc. [8]-[11]). The statistical behavior of 2D DFT coefficients is presented in [3]. The reconstruction algorithm based on variance behavior estimation for the case of 1D DFT is done in [1]. Herein we generalize these results for the case of 2D DFT, present two reconstruction algorithms and consider the reconstruction of ISAR images to illustrate the applicability of the presented approach.

The paper is organized as follows. Section II contains the basic theory survey regarding the CS and transform coefficients statistics in the case of signals with missing samples. In Section III we present the algorithms for the CS-based reconstruction. Numerical results are presented in Section IV, whereas the paper ends with concluding remarks.

II. BASIC THEORY

A. Compressed sensing and the 2D DFT domain

According to the compressive sensing theory, sparse signals can be reconstructed from a reduced set of observations. As the representative example, 2D DFT is observed as a domain of signal sparsity. Let us consider a $N \times M$ 2D signal K -sparse in the 2D DFT domain:

$$x(n, m) = \sum_{l=1}^K A_l e^{-j2\pi p_l n/N} e^{-j2\pi q_l m/M}, \quad (3)$$

where A_l denotes l -th component amplitude, whereas p_l and q_l denote frequencies of this component, belonging to the set $(p_l, q_l) \in \mathbf{K}$, $l = 1, \dots, K$. By definition, 2D DFT reads:

$$X(p, q) = \sum_{n=0}^{N-1} \sum_{m=0}^{M-1} x(n, m) e^{-j2\pi p n/N} e^{-j2\pi q m/M}, \quad (4)$$

$(n, m) \in \mathbf{N}$, $n = \{0, 1, \dots, N-1\}$, $m = \{0, 1, \dots, M-1\}$. If \mathbf{x} is used to denote the $NM \times 1$ vector of signal samples, whereas \mathbf{X} denotes the corresponding vector consisted of 2D DFT coefficients, then the 2D DFT transform can be written in matrix form as follows:

$$\mathbf{X} = \mathbf{W}\mathbf{x}. \quad (5)$$

The 2D DFT transform matrix is defined as:

$$\mathbf{W} = \mathbf{W}_{1D, N \times M} \otimes \mathbf{W}_{1D, N \times M} \quad (6)$$

that is, as the Kronecker product of the two corresponding 1D DFT transform matrices.

In the compressed sensing scenario, only $N_A M_A$ randomly positioned samples belonging to the set

$$(n_u, m_w) \in \mathbf{Q} \subseteq \mathbf{N}, \quad u = 1, 2, \dots, N_A, \quad w = 1, 2, \dots, M_A \quad (7)$$

are available, out of NM samples. Introducing partial 2D DFT transform matrix

$$\mathbf{A}_{cs} = \mathbf{\Phi}\mathbf{W}^{-1}, \quad (8)$$

containing rows corresponding to the positions of available samples ($\mathbf{\Phi}$ denotes random measurement matrix and \mathbf{W}^{-1} is the inverse 2D DFT matrix), the random selection of missing samples in the compressed sensing context can be modelled as:

$$\mathbf{x}_{cs} = \mathbf{A}_{cs}\mathbf{X} = \mathbf{\Phi}\mathbf{x}, \quad (9)$$

where the incoherence property of the matrix \mathbf{A}_{cs} is required in order to apply the linear programming in the CS-based reconstruction.

Furthermore, the CS-based reconstruction problem can be formulated as:

$$\min \|\mathbf{X}\|_1 \quad s.t. \quad \mathbf{x}_{cs} = \mathbf{A}_{cs}\mathbf{X}, \quad (10)$$

meaning that the aim is to find the sparsest possible solution under the condition that the constraints dictated by the available measurements set of equations remain satisfied. In the case when it is possible to determine the set of non-zero coefficients positions $\hat{\mathbf{p}}$ estimating the original set \mathbf{K} , the true values of transform coefficients can be determined as:

$$\mathbf{X}_r = \left(\mathbf{A}_{\hat{\mathbf{p}}}^H \mathbf{A}_{\hat{\mathbf{p}}} \right)^{-1} \mathbf{A}_{\hat{\mathbf{p}}}^H \mathbf{x}_{cs}. \quad (11)$$

where the matrix $\mathbf{A}_{\hat{\mathbf{p}}}$ is a submatrix of the measurement matrix \mathbf{A}_{cs} containing only columns corresponding to the set of estimated coefficients positions $\hat{\mathbf{p}}$. Operator $(\cdot)^H$ denotes the Hermitian transposition.

B. Statistical properties of 2D DFT transform coefficients

The respective mean values and variances of 2D DFT coefficients have been derived in [3], following the results for the 1D DFT case [2]. Therein, it has been shown that the influence of missing samples in measurements domain reflects to the corresponding transform domain representation similarly as an additive noise, whose statistical properties were determined. If \mathbf{X} denotes the matrix of initial 2D DFT, the non-signal coefficient mean value is equal to zero:

$$\mu_N(p, q) = E[X(p, q)] = 0, \quad (p, q) \notin \mathbf{K}, \quad (12)$$

whereas the variance can be represented as

$$\sigma_N^2(p, q) = \sum_{l=1}^K A_l M_A N_A \frac{MN - M_A N_A}{MN - 1}, \quad (p, q) \notin \mathbf{K}. \quad (13)$$

On the other side, the mean value of coefficients at signal positions have the non-zero mean value equal to:

$$\mu_s(p, q) = \sum_{l=1}^K A_l \delta(p - p_l, q - q_l). \quad (14)$$

The 2D DFT coefficient at the position $(p_s, q_s) \in \mathbf{K}$ has the variance equal to:

$$\sigma_s^2(p, q) = \sum_{l=1, l \neq s}^K A_l M_A N_A \frac{MN - M_A N_A}{MN - 1}, \quad (p, q) \in \mathbf{K}. \quad (15)$$

Comparing (13) and (15), we may conclude that the variances at non-signal and signal positions in the 2D DFT coefficient matrix differ. Namely, the variance of the coefficient at the signal position is reduced by the (scaled) amplitude value of the corresponding coefficient. This fact is the basic motivation for the reconstruction approach presented in this paper. If we are able to numerically approximate these variances, then we may define a criterion for the detection of non-zero transform coefficients, leading to a reconstruction approach for the missing samples, based on (11).

III. SIGNAL RECONSTRUCTION

A. Variance estimation

Based on the well-known robust theory, we may observe the loss function of the error in the 2D DFT coefficients calculation:

$$F[e_{p,q}(n, m)] = F\left[|x(n, m) e^{-j2\pi p n/N} e^{-j2\pi q m/M} - X(p, q)|\right] \quad (16)$$

where the function F corresponds to various norms. Namely, the general form assumes:

$$F[e_{p,q}(n, m)] = |e_{p,q}(n, m)|^L,$$

where the choice of the parameter L dictates different robustness to the influence of the external additive noise. Herein, the value $L = 2$ known as the maximum likelihood estimate of the full set of data and Gaussian noise. The total

error, for the randomly undersampled signal (9) can be expressed as follows:

$$GD(p, q) = \frac{1}{M_A N_A} \sum_{u=1}^{N_A} \sum_{w=1}^{M_A} F \left[x(n_u, m_w) e^{\frac{j2\pi p m_u}{N} - j\frac{2\pi q m_w}{M}} - \hat{X}(p, q) \right] \quad (17)$$

where $\hat{X}(p, q)$ represents the estimate of the signal's 2D DFT. For the observed case where $L = 2$ this estimate is obtained as:

$$\hat{X}(p, q) = \frac{1}{M_A N_A} \sum_{u=1}^{N_A} \sum_{w=1}^{M_A} x(n_u, m_w) e^{-j2\pi p m_u / N} e^{-j2\pi q m_w / M} \quad (18)$$

For different L other estimates are used [1]. Furthermore, it can be shown that for $L = 2$ function (17) becomes:

$$GD(p, q) = \text{var} \{ e_{p, q}(n_u, m_w) \}, u = 1, \dots, N_A, w = 1, \dots, M_A. \quad (19)$$

At the position of signal coefficient $(p, q) \in \mathbf{K}$ we have:

$$GD(p, q) = \sum_{l=1, l \neq s}^K A_l \frac{MN - M_A N_A}{MN - 1} < \sum_{l=1}^K A_l \frac{MN - M_A N_A}{MN - 1}, \quad (20)$$

meaning that $GD(p, q)$ can be used as an indicator whether the observed coefficient (p, q) belongs to the signal or not.

B. Reconstruction algorithms

Previous analysis can be used in the reconstruction of missing samples. If signal amplitudes $A_l, l = 1, 2, \dots, K$ have relatively close values, the reconstruction can be done with a simple non-iterative procedure. As it is previously published, the signal support can be determined by applying a threshold T on the calculated generalized deviations $GD(p, q)$. The threshold level can be calculated with respect to the maximal value of $GD(p, q)$, that is, $T = \alpha \max \{ GD(p, q) \}$ with α being a constant between 0.85 and 0.95, set heuristically [1]. The threshold is used to determine which positions (p, q) correspond to signal coefficients, as the variance estimated in $GD(p, q)$ is reduced when compared with non-signal positions. The reconstruction procedure is summarized as follows:

Algorithm 1: Non-iterative reconstruction

Input:

- Measurement vector \mathbf{x}_{cs}
- Measurement matrix \mathbf{A}_{cs}
- Constant parameter α

Calculations:

1. Calculate estimation $GD(p, q)$ according to (17) and (18)
2. $\hat{\mathbf{p}} \leftarrow \arg \{ GD(p, q) < T \}$
3. $\mathbf{A}_{\hat{\mathbf{p}}} \leftarrow \mathbf{A}_{cs}(:, \hat{\mathbf{p}})$
4. $\mathbf{X}_{\hat{\mathbf{p}}} \leftarrow (\mathbf{A}_{\hat{\mathbf{p}}}^H \mathbf{A}_{\hat{\mathbf{p}}})^{-1} \mathbf{A}_{\hat{\mathbf{p}}}^H \mathbf{x}_{cs}$
5. $\mathbf{X}_r \leftarrow \begin{cases} 0, & \text{for } k \notin \hat{\mathbf{p}} \\ X_{\hat{\mathbf{p}}}(k), & \text{for } k \in \hat{\mathbf{p}} \end{cases}$

Output:

- Reconstructed signal coefficients \mathbf{X}_r
-

In previous algorithm, $\mathbf{A}_{cs}(:, \hat{\mathbf{p}})$ is used to denote that only columns $\hat{\mathbf{p}}$ are used from the matrix \mathbf{A}_{cs} . Furthermore, if signal component amplitudes differ significantly, iterative version of the previous algorithm can be exploited in the reconstruction.

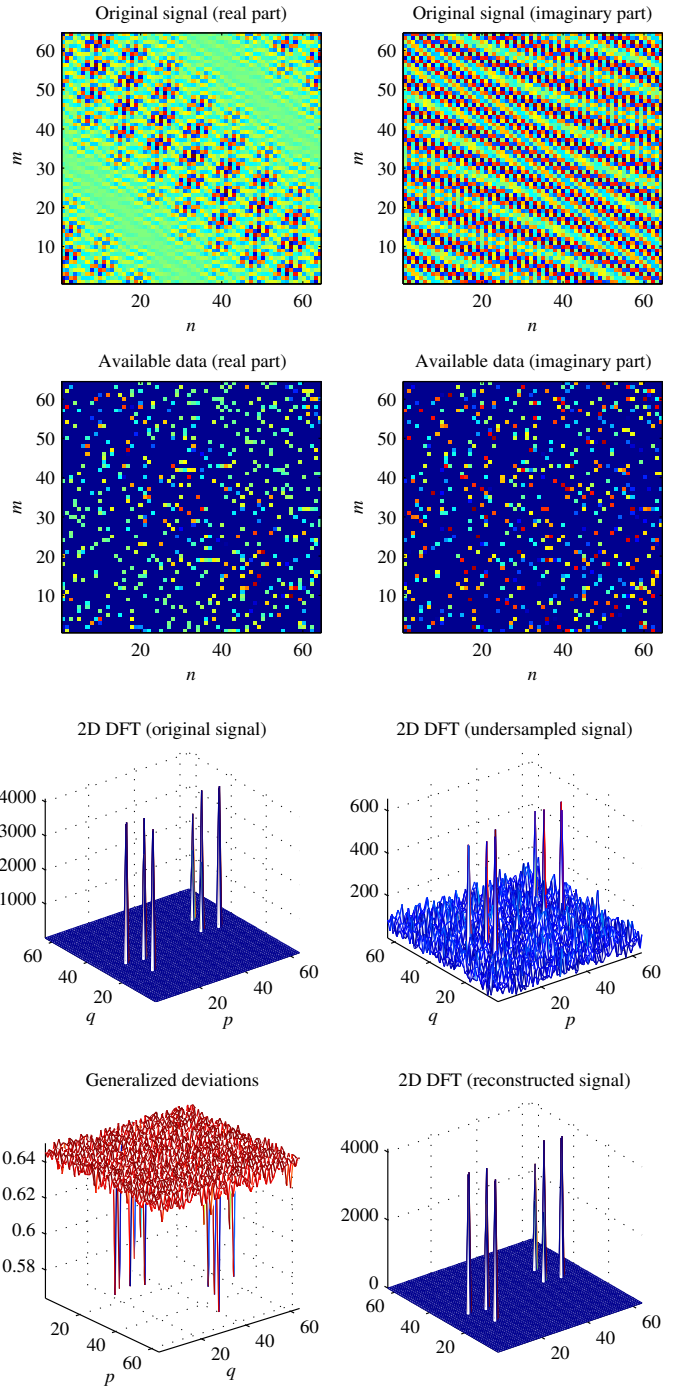


Fig. 1: The synthetic signal from example 1: first row: real and imaginary part of the original signal, second row: real and imaginary part of the available data, third row: 2D DFT of the original and undersampled signal; fourth row: $GD(p, q)$ calculated for the undersampled signal and the reconstructed 2D DFT

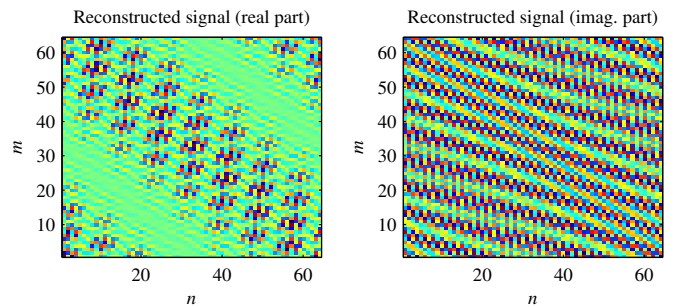


Fig. 2: Reconstructed signal from example 1 shown in the measurements domain

Namely, when a component at position (p_i, q_i) is detected as in previous algorithm, one can simply remove its contribution from the available measurements, and then repeat the same procedure for other components. The procedure can be repeated until a stopping criterion is met, e.g. when the energy of the residual after components removal becomes smaller than a fixed value ε . The procedure is described as follows:

Algorithm 2: Iterative reconstruction

Input:

- Measurement vector \mathbf{x}_{cs}
- Measurement matrix \mathbf{A}_{cs}
- Required precision ε

Calculations:

1. $\hat{\mathbf{p}} \leftarrow \emptyset$
2. $\mathbf{e} \leftarrow \mathbf{x}_{cs}$
3. **While** $\|\mathbf{e}\|_2^2 > \varepsilon$ **do**
4. Calculate $GD(p, q)$ (17) and (18) for \mathbf{e}
5. $(\hat{p}, \hat{q}) \leftarrow \arg \min GD(p, q)$
6. $\hat{\mathbf{p}} \leftarrow \hat{\mathbf{p}} \cup (\hat{p}, \hat{q})$
7. $\mathbf{A}_{\hat{\mathbf{p}}} \leftarrow \mathbf{A}_{cs}(:, \hat{\mathbf{p}})$
8. $\mathbf{X}_{\hat{\mathbf{p}}} \leftarrow (\mathbf{A}_{\hat{\mathbf{p}}}^H \mathbf{A}_{\hat{\mathbf{p}}})^{-1} \mathbf{A}_{\hat{\mathbf{p}}}^H \mathbf{x}_{cs}$
9. $\mathbf{y}_{\hat{\mathbf{p}}} \leftarrow \mathbf{A}_{\hat{\mathbf{p}}} \mathbf{X}_{\hat{\mathbf{p}}}$
10. $\mathbf{e} \leftarrow \mathbf{x}_{cs} - \mathbf{y}_{\hat{\mathbf{p}}}$
11. **end while**
12. $\mathbf{X}_r \leftarrow \begin{cases} 0, & \text{for } k \notin \hat{\mathbf{p}} \\ X_{\hat{\mathbf{p}}}(k), & \text{for } k \in \hat{\mathbf{p}} \end{cases}$

Output:

- Reconstructed signal coefficients \mathbf{X}_r
-

Previously described algorithm can be enhanced by combining with the non-iterative procedure, in sense that coefficients blocks can be detected in every iteration.

IV. EXPERIMENTAL RESULTS

Example 1: Let us observe a sparse signal in the 2D HT domain. Signal size is $M \times N = 64 \times 64$ and only 596 samples are available (14.6%) at random positions. Real and imaginary parts of the signal are shown in Fig. 1: first row, whereas available data are shown in Fig. 1, second row. Dark blue parts denote positions of unavailable samples. Absolute values of the 2D FT of the signal with all samples available and the undersampled signal are shown in Fig. 1, third row.

The function $GD(p, q)$ was calculated according to (17) and it is shown in Fig. 1, fourth row (left). The Algorithm 1 was exploited in the successful reconstruction of the signal, and its output is shown in Fig. 1, fourth row (right). The reconstructed signal in the measurement domain is shown in Fig 2.

Example 2: Commonly used MIG 25 test ISAR data is considered in this example [4]. It has 64 pulses with 64 samples within each pulse. As the target motion was not completely compensated and the motion was not completely uniform, we sparsified the signal by setting smallest coefficients to zero before further processing. Various percents

of missing samples and pulses at random positions are considered: 29%, 49% and 68%.

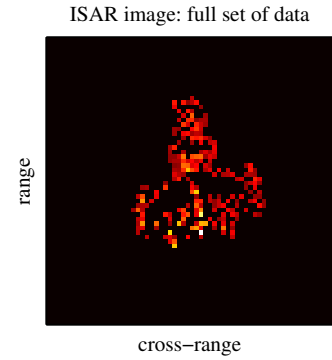


Fig. 3: ISAR image of MIG 25, obtained based on the full set of data

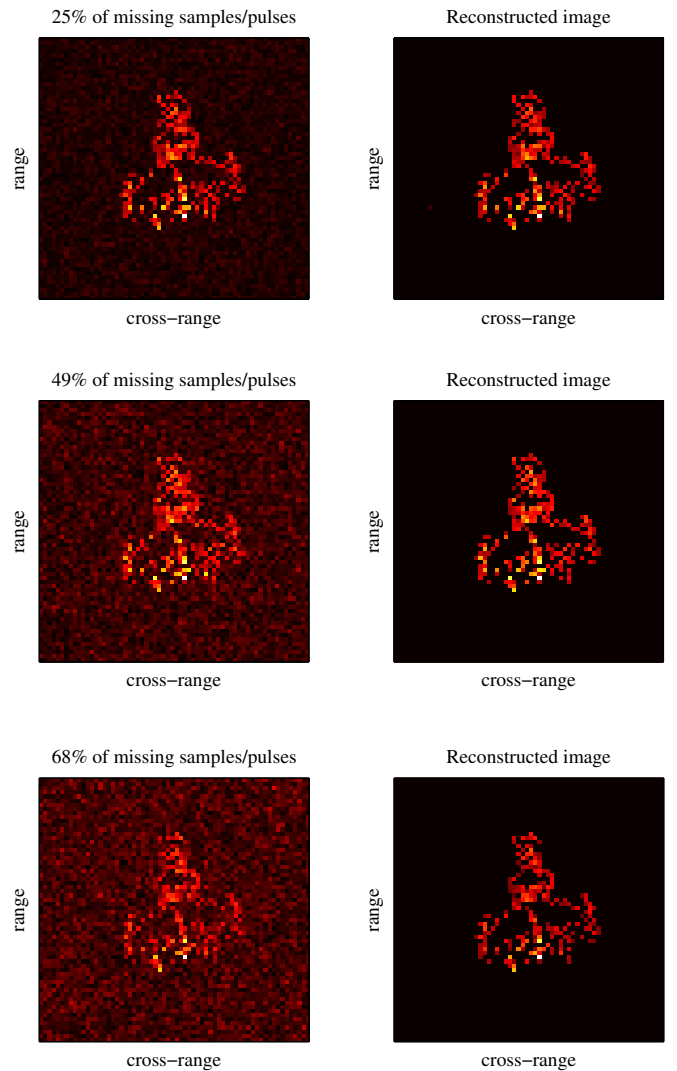


Fig. 5: The ISAR image reconstruction - MIG 25 example. The reconstruction is performed based on signals having 25%, 49%, and 68% of samples/pulses unavailable. Images without reconstruction are shown left, whereas the reconstructed images are shown right.

ISAR image obtained based on the full set of data is shown in Fig. 3., whereas the reconstruction was performed based on Algorithm 2 and the results are shown in Fig. 4. The ISAR images obtained directly from the available samples (with all missing samples set to zero) are shown on the left side.

V. CONCLUSION

In this paper, an efficient method for the compressed sensing based reconstruction of 2D signals having missing samples is presented. Based on the transform coefficients variance for randomly undersampled signals, a criterion for signal coefficients detection is presented. It is the basic step in simple reconstruction algorithms. First algorithm is non-iterative, and it is suitable for the recovery of signals having components with similar amplitude values. The iterative form is a general reconstruction algorithm. The second algorithm can be easily generalized such that blocks of coefficients are recovered, instead of a single component in every iteration.

These algorithms have a potential for the robust reconstruction of undersampled signals in impulsive noise environments. Moreover, by modifying a parameter in the calculation of generalized deviations, its applicability can be extended for other types of noises.

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