A Resistive Circuits Analysis Using Graph Spectral Decomposition

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***Abstract*—A method for a resistive circuit analysis based on graph spectral decompositions is proposed. It is shown that the Laplacian matrix can be used in order to calculate node potentials. Based on the Laplacian eigenvalues and eigenvectors it is possible to decompose a complex resistive circuit into smaller, weakly connected sub-circuits.**

***Keywords*—Graph spectra; Electrical networks; Resistive circuits; Spectral decomposition; Laplacian**

I. Introduction

Graph theory has been used in a wide variety of problems including electrical networks, social networks, machine learning, communication networks, signal and image processing.

An electrical network is frequently modeled as a collection of interconnected two-pole components. This model corresponds to a weighted graph where edges correspond to individual circuit components, and edge weights correspond to the component parameters. There exist many techniques for the circuit analysis based on graph incidence matrices, node voltages, graph trees, and independent contours [1].

Spectral graph theory [2] is an emerging field of graph theory, theoretically developed over the past 60 years with many recently developed applications [3]–[6]. In [3], the graph Laplacian is used to perform the Kron reduction of the electrical network. The effective graph resistance and closed-form solution for equivalent circuit resistance are developed in [4], [5]. Signal processing on graphs is reviewed in [6]. Graph spectrum is obtained by eigen-decomposition of the corresponding Laplacian matrix [2], [6], [7].

In this paper, we will consider a resistive electrical network. It is shown that network voltages and currents can be calculated by using the spectral decomposition of the graph Laplacian matrix.

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The proposed method is demonstrated on two example circuits. It will be shown that the spectral decomposition can be used to identify an (almost) independent part of the circuit (a sub-circuit with weak connections to the remaining part of the network). In this case, the Laplacian eigenvectors are concentrated on a subset of nodes that belongs to the considered weakly connected sub-circuit.

II. Proposed approach

Let us consider a passive resistive electric circuit with N nodes and N_b branches. For connecting nodes n and m, denote the branch resistance with R_{nm}. The circuit can be represented as a weighted graph, where the edge weights are conductances $w_{nm} = 1/R_{nm}$.

The graph Laplacian can be obtained as $\mathbf{L} = \mathbf{D} - \mathbf{W}$ where $\mathbf{W}$ is an edge weight matrix with elements $w_{nm} = 1/R_{nm}$ when there is a branch between the node n and the node m and $w_{nm} = 0$ otherwise. $\mathbf{D}$ is a diagonal matrix with $d_{nn} = \sum_{m=1}^{N} w_{nm}$. Laplacian matrix eigenvalues are denoted with λ_k and their corresponding eigenvectors with $\mathbf{u}_k$.

If the corresponding graph is connected, there is only one zero eigenvalue $\lambda_1 = 0$ of the Laplacian matrix, and all other eigenvalues are positive. Here we will assume that eigenvalues are sorted in non-decreasing order $\lambda_k \leq \lambda_{k+1}$ for $k = 1, 2, \dots, N-1$. The eigenvector $\mathbf{u}_1$ corresponding to the $\lambda_1 = 0$ is constant.

$$\mathbf{u}_1 = \frac{1}{\sqrt{N}}[1, 1, \dots, 1]^T$$

If we arrange eigenvalues into a diagonal matrix $\mathbf{\Lambda}$ and eigenvectors into a square matrix $\mathbf{U} = [\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_N]$, then we can write

$$\mathbf{L} = \mathbf{U}\mathbf{\Lambda}\mathbf{U}^T \tag{1}$$

Note that the matrix $\mathbf{U}$ is unitary, $\mathbf{U}\mathbf{U}^T = \mathbf{E}$ where $\mathbf{E}$ is a unity matrix with ones on the main diagonal and zeros elsewhere.

Assume that node 1 is a reference node and that there exist external current generators connected between reference node 1 and nodes $2, 3, \dots, N$. Now we can form an external currents vector $\mathbf{i}$ as

$$\mathbf{i} = [i_1, i_2, \dots, i_N]^T$$

where $i_1 = -i_2 - i_3 - \dots - i_N$, according to Kirchhoff's first law.

The potential of node n will be denoted with v_n. The potential vector $\mathbf{v}$ is

$$\mathbf{v} = [v_1, v_2, \dots, v_N]^T$$

According to Kirchhoff's first law for each node n, the sum of all currents should be equal to the external current i_n. The current at a branch that connects nodes n and m is equal to

$$i_{nm} = (v_n - v_m)/R_{nm} = (v_n - v_m)w_{nm}$$

We can write Kirchhoff's first law for node n as

$$i_n = \sum_{m=1}^{N} i_{nm} = \sum_{m=1}^{N} (v_n - v_m)w_{nm}$$

Note that $w_{nm} = 0$ if there is no branch between node n and node m. This relation can be rewritten as

$$i_n = v_n \sum_{m=1}^{N} w_{nm} - \sum_{m=1}^{N} v_m w_{nm}$$

or in a matrix form as

$$\begin{aligned} \mathbf{i} &= \mathbf{Dv} - \mathbf{Wv} = (\mathbf{D} - \mathbf{W})\mathbf{v} \\ \mathbf{i} &= \mathbf{Lv} \end{aligned} \tag{2}$$

This relation can be considered as Ohm's law applied to a whole circuit, where conductances are included in the Laplacian matrix $\mathbf{L}$. In classical circuit analysis, the system of equations given by (2) is known as the vertex potentials equation.

Vectors $\mathbf{i}$ and $\mathbf{v}$ can be considered as signals defined on a given graph. A spectral representation of these signals is obtained as their projection on the Laplacian eigenvectors

$$\begin{aligned} \mathbf{I} &= \mathbf{U}^T\mathbf{i} \\ \mathbf{V} &= \mathbf{U}^T\mathbf{v} \end{aligned} \tag{3}$$

According to (1), (2) and (3) we can write

$$\begin{aligned} \mathbf{i} &= \mathbf{U\Lambda U}^T\mathbf{v} \\ \mathbf{U}^T\mathbf{i} &= \mathbf{\Lambda U}^T\mathbf{v} \\ \mathbf{I} &= \mathbf{\Lambda V} \end{aligned} \tag{4}$$

The relation (4) can be considered as Ohm's law in the spectral domain. The eigenvalue matrix $\mathbf{\Lambda}$ is diagonal, resulting in

$$I_k = \lambda_k V_k \tag{5}$$

for $k = 1, 2, \dots, N$.

Eqn. (5) can be solved for V_k for each k, except for $k = 1$

$$V_k = \frac{1}{\lambda_k} I_k$$

For $k = 1$, we have $\lambda_1 = 0$ resulting in $I_1 = 0$ and arbitrary V_1. Note that $I_1 = \mathbf{u}_1^T\mathbf{i} = 0$ according to Kirchhoff's first law.

The value of V_1 can be defined if we state that the reference node potential v_1 is zero. From (3), we can write

$$\mathbf{v} = \mathbf{UV}$$

and

$$v_1 = u_1(1)V_1 + u_2(1)V_2 + \dots + u_N(1)V_N$$

From this equation we can find V_1, having in mind that $v_1 = 0$ as

$$V_1 = -\frac{u_2(1)V_2 + u_3(1)V_3 + \dots + u_N(1)V_N}{u_1(1)}$$

III. Examples

We will illustrate the proposed approach in two examples. A simple circuit is analyzed in Example 1. A more complex circuit, composed of two weakly connected subcircuits, is analyzed in Example 2.

A. Example 1

Let us consider the circuit presented in Fig. 1. The corresponding weighted graph is presented in Fig. 2.

The graph Laplacian is

$$\mathbf{L} = \frac{1}{24000}\begin{bmatrix} 12 & -6 & 0 & -6 & 0 & 0 & 0 \\ -6 & 19 & -4 & -6 & -3 & 0 & 0 \\ 0 & -4 & 16 & 0 & -12 & 0 & 0 \\ -6 & -6 & 0 & 24 & -6 & -6 & 0 \\ 0 & -3 & -12 & -6 & 27 & -6 & 0 \\ 0 & 0 & 0 & -6 & -6 & 36 & -24 \\ 0 & 0 & 0 & 0 & 0 & -24 & 24 \end{bmatrix}$$

The eigenvalues of the Laplacian are

$$\begin{aligned} \lambda_1 &= 0 \\ \lambda_2 &= 0.251 \times 10^{-3} \\ \lambda_3 &= 0.406 \times 10^{-3} \\ \lambda_4 &= 0.956 \times 10^{-3} \\ \lambda_5 &= 1.091 \times 10^{-3} \\ \lambda_6 &= 1.542 \times 10^{-3} \\ \lambda_7 &= 2.337 \times 10^{-3} \end{aligned}$$

and corresponding eigenvectors are presented in Fig. 3.

The vector of external currents is given by

$$\mathbf{i}_g = 10^{-3} \times \begin{bmatrix} -7 & 0 & 0 & 0 & 0 & 0 & 7 \end{bmatrix}^T$$

The vector of node potentials is given by

$$\mathbf{v} = \begin{bmatrix} 0 & 12 & 18 & 16 & 20 & 32 & 39 \end{bmatrix}^T$$

and in the spectral domain it is equal to

$$\mathbf{V} = \begin{bmatrix} -51.78 & 30.28 & -7.35 & 3.41 & 1.29 & 0.74 & 1.79 \end{bmatrix}^T$$

B. Example 2

Let us consider a more complex circuit shown in Fig. 4.

The weighted circuit graph is presented in Fig. 5. The Laplacian eigenvalues are:

$$\begin{aligned}
\lambda_1 &= 0 \\
\lambda_2 &= 0.001 \times 10^{-3} \\
\lambda_3 &= 0.251 \times 10^{-3} \\
\lambda_4 &= 0.398 \times 10^{-3} \\
\lambda_5 &= 0.406 \times 10^{-3} \\
\lambda_6 &= 0.956 \times 10^{-3} \\
\lambda_7 &= 0.984 \times 10^{-3} \\
\lambda_8 &= 1.091 \times 10^{-3} \\
\lambda_9 &= 1.341 \times 10^{-3} \\
\lambda_{10} &= 1.543 \times 10^{-3} \\
\lambda_{11} &= 1.778 \times 10^{-3} \\
\lambda_{12} &= 2.338 \times 10^{-3}
\end{aligned}$$

We can see that eigenvalue λ_2 is very close to zero. The Laplacian eigenvectors are shown in Fig. 6.

Form Fig. 6, we can observe that eigenvectors 4, 7, 9 and 11 have zero values at nodes 1–7, while eigenvectors 3, 5, 6, 8, 10 and 12 have zero values at nodes 8–12.

Eigenvectors 1 and 2 span over all nodes. Corresponding eigenvalues are almost equal, $\lambda_2 \approx \lambda_1 = 0$, meaning that these two eigenvectors belong to the same sub-space determined with a zero eigenvalue. Hence, we can use their linear combination

$$\begin{aligned}
\mathbf{u}_1^{new} &= a_1\mathbf{u}_1 + a_2\mathbf{u}_2 \\
\mathbf{u}_2^{new} &= b_1\mathbf{u}_1 + b_2\mathbf{u}_2
\end{aligned}$$

with

$$\begin{aligned}
a_1^2 + a_2^2 &= 1 \\
b_1^2 + b_2^2 &= 1 \\
a_1 b_1 + a_2 b_2 &= 0
\end{aligned}$$

We can select parameters a_1, a_2, b_1 and b_2 such that we obtain eigenvectors $\mathbf{u}_1^{new}$ and $\mathbf{u}_2^{new}$ spanned over nodes 1–7 and 8–12. Orthonormality of the matrix $\mathbf{U}$ is preserved. For the considered case $a_1 = -b_2 \approx \sqrt{\frac{5}{12}}$, and $a_2 = b_1 \approx \sqrt{\frac{7}{12}}$.

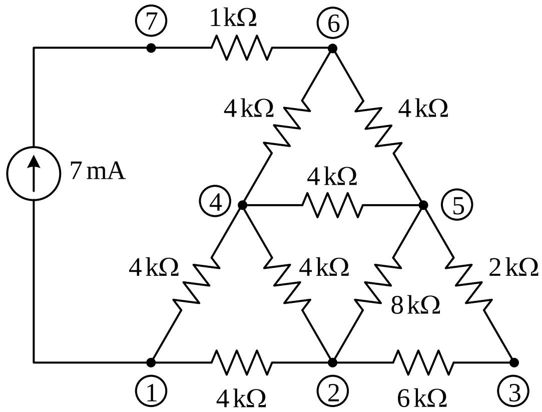

Fig. 1. A circuit from Example 1

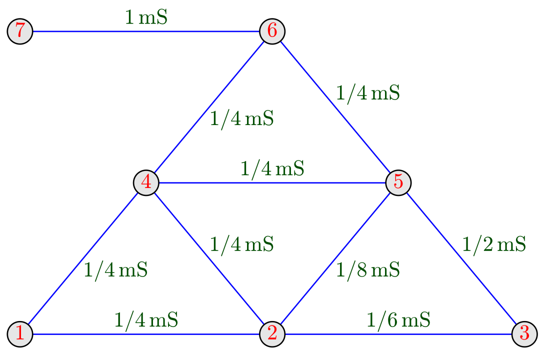

Fig. 2. The weighted graph of the circuit shown in Fig. 1

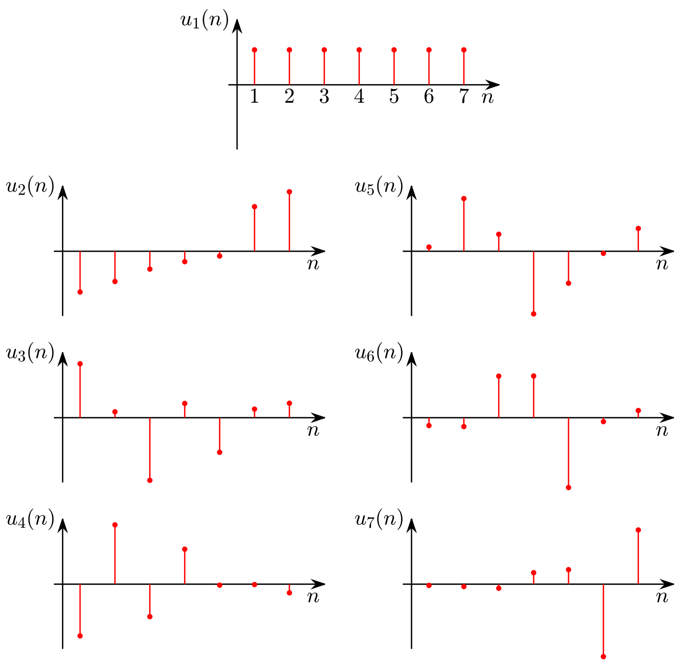

Fig. 3. Laplacian eigenvectors from Example 1

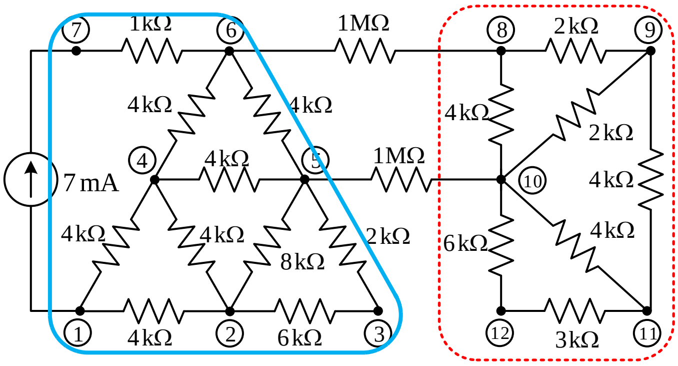

Fig. 4. A circuit from Example 2

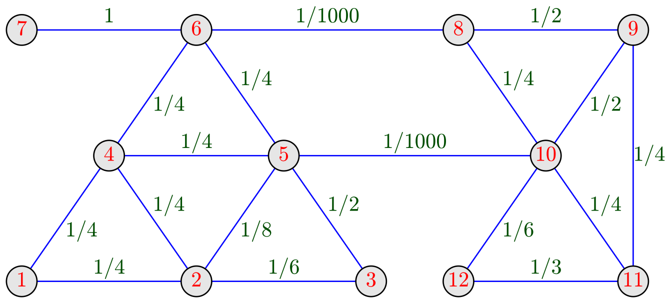

Fig. 5. The weighted graph for the circuit shown in Fig. 4, edge weights are in mS

In this way, based on the spectral decomposition, we can conclude that the analyzed circuit may be split into two, almost independent, parts.

IV. Conclusion

We studied a possible application of the graph spectral theory in resistive electrical network analysis and decomposition. It is shown that the graph Laplacian relates node potentials and external node circuits for a resistive network. A similar relation is derived in the graph spectral domain. These two relations can be considered as "Ohm's law" for a whole graph in physical and spectral domains.

By analyzing graph spectra, we can check if the considered graph (circuit) is strongly connected, or if there exist subgraphs (sub-circuits) that are weakly connected with the rest of the graph.

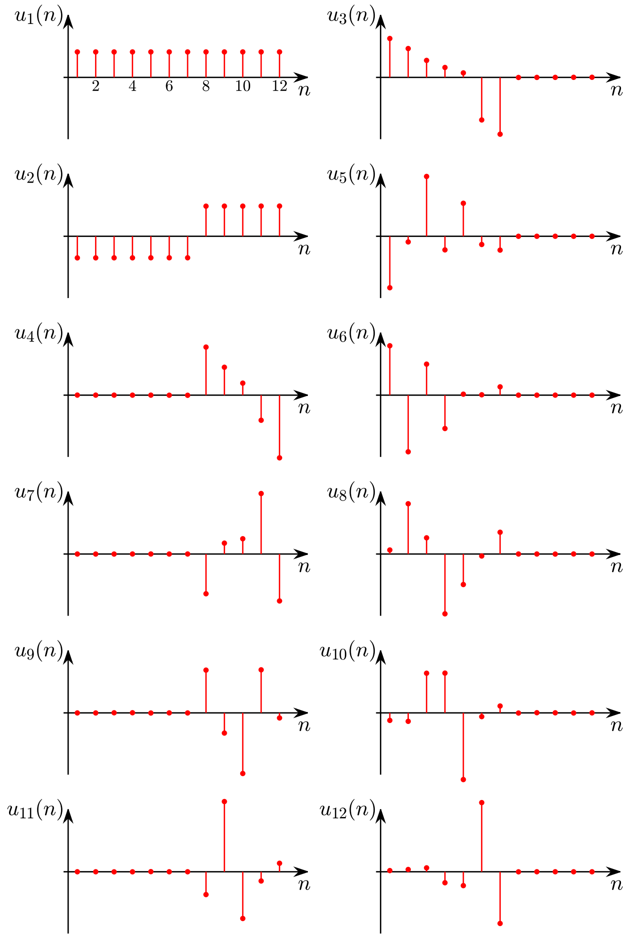

Fig. 6. Laplacian eigenvectors from Example 2

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