

Sparse signal reconstruction using gradient-threshold based method

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Abstract— The performance of gradient (steepest descent) and the threshold-based algorithms are observed in terms of the sparse signal reconstruction. The advantages of both methods are combined within the new approach used to recover all samples from randomly under-sampled signal. The gradient-based algorithm may fail to recover the signal unless a relatively large number of iterations is performed, which can be time consuming. The procedure can be speed up by stopping the gradient algorithm at certain convenient iteration and continuing with the reconstruction using the threshold-based method. Threshold is calculated in a way to separate the signal components from the spectral noise that is still left in the signal after the gradient-based reconstruction. The exact values of the signal amplitudes are calculated by solving the optimization problem. The proposed method increases the reconstruction speed with satisfactory reconstruction accuracy. The theory is proved with experiments.

Keywords— compressive sensing; gradient based algorithm; threshold; variance

I. INTRODUCTION

There is a constant intent to reduce the amount of data in many real applications. Lowering the number of signal samples during the signal acquisition is topic of popular theory known as the Compressive Sensing (CS) [1]-[11]. The CS allows under-sampling and later signal reconstruction from far less samples compared to the traditional approach based on the Shannon-Nyquist sampling theorem. The possibility to apply the CS approach could be limited by the signal nature and the acquisition process, but still the application field of the CS is wide.

In order to recover the signal from a small number of collected (or available) samples [10]-[26], different mathematical optimization procedures can be used. Some of them are fast, such as greedy approaches [13], [15], [17], but these are not always accurate. Some other approaches, such as convex optimization [3], [26], are more accurate but computationally demanding and thus time consuming.

In this paper, we consider the two approaches, one belonging to the convex optimization group and the other belonging to the greedy methods, with an aim to combine them and to exploit the advantages of both. The first approach is called the gradient-based approach (GA) [14], [16], [20],

[26], which allows reconstruction from a small set of samples through the iterative update of missing samples values. Since it assumes nested iterations, the algorithm execution could be time consuming especially when reconstruction accuracy is the priority. The second approach is the threshold-based algorithm used to separate the signal components from the noise appearing in the spectral domain as a result of the missing samples [3], [6], [19], [22]. This approach is faster compared to the GA but fails to detect positions of the signal components if there are significant variations between the values of components amplitudes, or if the number of missing samples is large.

Therefore, in this paper we combined the advantages of both methods. The gradient in GA is observed for various signals and it is noticed that it becomes larger than noise after just a few iterations. In the original procedure, the algorithm step is decreased in that point and the algorithm is continued with the new step. However, in the proposed procedure the GA is stopped here and the threshold is applied in order to eliminate the noise remained in the signal. The threshold is defined by using the variance of the gradient in the stopping point. Namely, the threshold and gradient are combined in the proposed method, in order to speed up the reconstruction procedure, without degrading the reconstruction accuracy. The positions of the signal components in the spectral domain are revealed after the threshold is applied. The procedure involves an optimization problem solving in order to recover the values of the components' amplitudes.

II. THEORETICAL BACKGROUND

A. Sparse signal reconstruction

According to the CS theory, the signal can be reconstructed from a small set of randomly selected samples if there is a certain transformation domain where the signal has a sparse or compact representation. We started with the assumption that signal $s(n)$ of length N is sparse in transform domain $S(k)=T[s(n)]$ with sparsity K , ($K \ll N$), where T in this paper denotes the Discrete Fourier Transform (DFT) operator. If we assume that only M ($K < M$) samples are available, the following relation holds [3]:

$$\mathbf{s}_a = \mathbf{A}\mathbf{S}, \quad (1)$$

where \mathbf{s}_a is the measurement vector and \mathbf{A} is the CS matrix formed as $\mathbf{A}=\mathbf{\Phi}\mathbf{T}^{-1}$. Matrices $\mathbf{\Phi}$, \mathbf{T} and \mathbf{T}^{-1} are the measurement matrix, the DFT and the inverse DFT matrix

respectively. The sparsest solution of the undetermined system of equations (1) can be found by counting the number of nonzero elements using ℓ_0 norm. This is an NP-hard combinatorial optimization problem and therefore, the ℓ_1 norm is more often used:

$$\min \|\mathbf{S}\|_{\ell_1} \text{ subject to } \mathbf{s}_a = \mathbf{A}\mathbf{S}, \quad (2)$$

where $\|\mathbf{S}\|_{\ell_1} = \sum_{k=0}^{N-1} |S(k)|$. There is a number of algorithms for sparse signal reconstruction [1]-[3], [9], [11]-[13]. In the sequel, two commonly used algorithms are observed: gradient based algorithm [26] and single iteration reconstruction algorithm [3],[6].

B. Gradient based algorithm

The gradient-based algorithm belongs to the group of convex optimization algorithms, where missing samples are considered as the variables with the zero initial value. It is iterative algorithm where in each iteration the value of missing sample is changed for $+\Delta$ and $-\Delta$, approaching its exact value [14], [16], [20], [22].

Before applying the algorithm, the initial signal is defined:

$$s^{(0)}(n) = \begin{cases} s_a(n) & \text{for available samples, } n \in \{n_1, n_2, \dots, n_M\} \\ 0 & \text{for missing samples, } n \notin \{n_1, n_2, \dots, n_M\} \end{cases} \quad (3)$$

Objective of gradient-based algorithm is to determine the value of the gradient which is used to update the values of the missing samples. In $iter$ -th iteration, the gradient vector $Grad(i)$ is estimated by using the finite differences of the sparsity measure. It is calculated for each missing sample at instants $i \notin \{n_1, n_2, \dots, n_M\}$ which indicates the missing samples positions:

$$Grad_{iter}(i) = \left(\|\mathbf{S}_g^+\|_{\ell_1} - \|\mathbf{S}_g^-\|_{\ell_1} \right) / (2\Delta N), \quad (4)$$

where

$$\mathbf{S}_g^+ = \mathbf{T}\mathbf{s}^+, \quad \mathbf{S}_g^- = \mathbf{T}\mathbf{s}^-. \quad (5)$$

On the other hand,

$$s^+(n) = s^{(iter)}(n) + \Delta\delta(n-i) \quad (6)$$

$$s^-(n) = s^{(iter)}(n) - \Delta\delta(n-i),$$

$$\begin{aligned} |S_g^+(k)| &= |S_g^{(iter)}(k) + \Delta D_i(k)| \\ |S_g^-(k)| &= |S_g^{(iter)}(k) - \Delta D_i(k)| \end{aligned} \quad (7)$$

where $S_g^{(iter)}(k) = T[s^{(iter)}(n)]$ and $D_i(k) = T[\delta(n-i)]$. At the positions of the available signal samples the gradient vector is zero.

III. A SPARSE SIGNAL RECONSTRUCTION PROCEDURE BASED ON GRADIENT AND THRESHOLD

In the sequel, the proposed procedure for sparse signal reconstruction based on the gradient algorithm and threshold calculation is described. After several iterations of gradient algorithm, the gradient estimate starts to oscillate around the stationary point. The mean squared error becomes almost constant and it is not possible to achieve an improvement in

the signal concentration for a chosen value of step Delta (Δ). In other words, the best possible sparsity with chosen Δ is obtained. In original procedure, the execution of the algorithm can be iteratively continued with reduced value of Δ [26]. Having in mind that the iterations are time-consuming, we proposed a modification of the original procedure in order to speed it up. The algorithm is stopped in this point and the appropriate threshold for removing the remaining spectral noise is calculated. The threshold that separated noise from signal components in the spectral domain can be calculated as follows [6]:

$$T_1 = \sqrt{-\sigma^2 \log(1-P(T)^{1/N})}, \quad (8)$$

where P is the probability that noise is lower than T_1 threshold (it is set to $P(T)=0.99$ [3], [6], [18]), σ denotes variance of noise caused by missing samples:

$$\sigma^2 = [M(N-M)/(N-1)] \sum_{i=1}^K A_i^2. \quad (9)$$

The parameter A_i denotes the amplitude of the i -th signal component. However, when the amplitudes of signal are very small, the threshold T_1 cannot separate signal components from noise. In order to overcome this problem, GA is applied first, until algorithm starts to oscillate what happens after few iterations. Variance of noise left in signal, originating from incorrect signal values, becomes below the gradient value. Therefore, new threshold that would detect signal component above the noise should be defined. The expression for the new threshold T_2 should include the variance of the gradient, which is also the maximum value of the noise variance in the signal spectrum, multiplied by scaling factor $S_f = [M(N-M)/(N-1)]$:

$$T_2 = \sqrt{-S_f \sigma_{Grad}^2 \log(1-P(T)^{1/N})}. \quad (10)$$

The variance of gradient on missing samples is calculated using the gradient value, and number of missing samples:

$$\sigma_{Grad}^2 = \frac{1}{N-M} \sum_{i=1}^{N-M} \left| Grad(i) - \frac{1}{N-M} \sum_{i=1}^{N-M} Grad(i) \right|^2. \quad (11)$$

Then the positions of the signal components are detected by applying T_2 on the vector \mathbf{S}_g :

$$\mathbf{p} = \arg \max \{ |\mathbf{S}_g| > T_2 \}. \quad (12)$$

The measurement vector is defined as $\mathbf{s}_a = \mathbf{A}\mathbf{S}_g$, where \mathbf{A} denotes the CS matrix formed by using columns of DFT matrix that correspond to the frequencies \mathbf{p} of signal components and rows corresponding to M available measurements. The vector \mathbf{S}_g is the DFT vector obtained in the iteration before Δ is changed. The least square optimization problem is solved in order to recover the components' amplitudes. The amplitudes of the signal components whose positions are in vector \mathbf{p} , and recovered DFT vector \mathbf{S} , are obtained as a solution of the following problem [3], [6], [18]:

$$\mathbf{S} = (\mathbf{A}^* \mathbf{A})^{-1} (\mathbf{A}^* \mathbf{s}_a). \quad (13)$$

IV. EXPERIMENTAL RESULTS

Example 1: The sinusoidal signal with length of 256 samples

and five components with different amplitudes is considered. The signal is sparse in the DFT domain and has the following form:

$$a(n) = A_1 \cos(32\pi n / N) + A_2 \cos(64\pi n / N) + A_3 \cos(128\pi n / N) + A_4 \cos(192\pi n / N) + A_5 \cos(112\pi n / N); \quad (14)$$

where $A_1=3.5$, $A_2=3$, $A_3=1$, $A_4=4$, $A_5=2$ are amplitudes of the signal components. The DFT of the signal is shown in Fig 1. Signal has 62% of the missing samples. The initial DFT of the signal is shown in Fig. 2a. In order to detect components positions, the first step is applying the GA reconstruction on the initial DFT, until GA reaches its stationary point (in this example after 13 iterations).

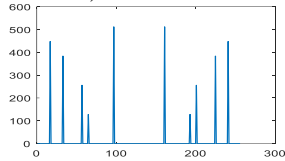


Figure 1. The DFT of the signal defined by (14)

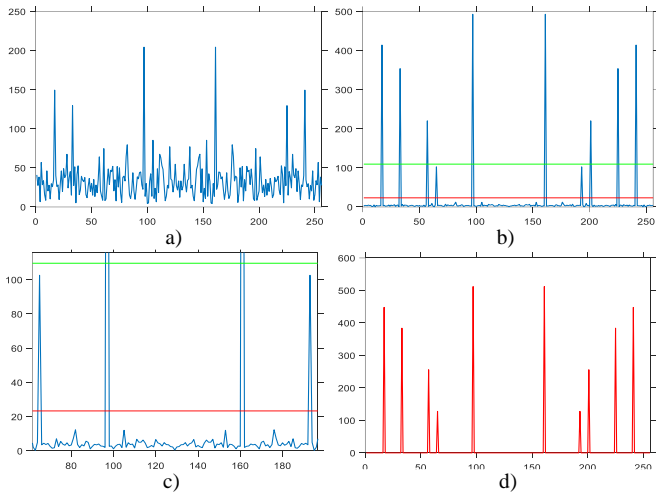


Figure 2. a) The initial Fourier Transform with 62% missing samples; b) The reconstructed Fourier Transform using 13 iterations of GA; c) Zoomed part of Fig. 1b); d) The reconstructed DFT by using the proposed threshold

The signal recovered after GA is shown in Fig. 2b. It is important to emphasize that, after this step, the signal amplitudes are not reconstructed exactly, and a certain percent of noise is left in the signal (noise can be seen from the zoomed DFT region in Fig. 2c).

Applying the T_1 threshold on the reconstructed DFT, in order to remove the noise left in the signal, failed in the detection of all signal components due to their variable amplitudes. The T_1 threshold is shown with green line in Fig. 2b. Therefore, the calculation of the threshold T_2 is done in this step (T_2 is shown in Fig. 2b with red line). It can be seen that T_2 successfully detects all signal components, and at the same time, keeps the spectral noise below the level of the smallest signal component. After noise removal, the amplitudes of the signal components are recovered by an optimization problem solving. The reconstructed signal is shown in Fig. 2d.

Example 2: Let us now test the proposed procedure on the another signal consisted of more components. The signal has 8 components, 256 samples length and has the following form:

$$a(n) = A_1 \cos(16\pi n / N) + A_2 \cos(20\pi n / N) + A_3 \cos(28\pi n / N) + A_4 \cos(32\pi n / N) + A_5 \cos(64\pi n / N) + A_6 \cos(128\pi n / N) + A_7 \cos(192\pi n / N) + A_8 \cos(112\pi n / N); \quad (15)$$

The amplitudes of the components are $A_1=1.20$, $A_2=1.10$, $A_3=1.30$, $A_4=1.50$, $A_5=1.40$, $A_6=0.90$, $A_7=0.39$, $A_8=0.47$, respectively, and the number of measurements used for reconstruction is $M=96$.

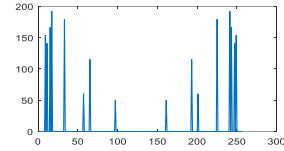


Figure 3. The DFT of the signal defined by (15)

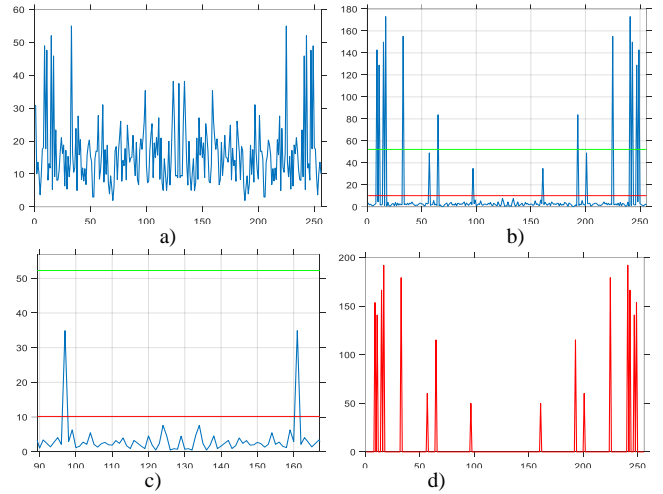


Figure 4. a) The initial DFT with 62% missing samples, b) The reconstructed DFT using 16 iterations of GA, c) Zoomed part of Fig. 1b) with the smallest signal amplitude d) The reconstructed DFT based on the proposed threshold

The signal DFT is shown in Fig. 3, while the initial DFT calculated by using 38% of the signal samples is shown in Fig. 4a. The first step is the detection of the components positions, by applying the GA until it reaches the stationary point (after 16 iterations in this example). We tested the possibility to detect components positions by applying the threshold T_1 , but it failed as it can be seen from Fig. 4b (T_1 is marked with green line). Therefore, T_2 is calculated and succeeded in the detection of the positions of all components. The threshold T_2 is shown in Fig. 4b with red line, while the reconstructed DFT of the signal is shown in Fig. 4d.

The processing speed of the proposed algorithm and the speed of GA (in a full number of iterations) are measured and compared. The results are given in Table 1. The time required for execution of the proposed algorithm is much smaller compared to the time required for GA execution.

As it was already mentioned, the GA with a decreased number of iterations can detect the components positions but fails in the reconstruction of the components amplitudes. The

values of components amplitudes after the GA and the proposed method reconstruction, are shown in Table 2 (the amplitudes are scaled by $N/2$). It can be seen that the amplitudes reconstructed with the proposed method, correspond to the exact amplitudes, while GA reconstruction introduces certain error.

TABLE I. EXECUTION TIMES FOR PROPOSED ALGORITHM AND GA IN FULL NUMBER OF ITERATIONS

Example 1		Example 2	
Time elapsed [s]		Time elapsed [s]	
The proposed method (13 iterations of GA and threshold T_2)	Full number of GA iterations (60 iterations)	The proposed method (16 iterations of GA and threshold T_2)	Full number of GA iterations (100 iterations)
0.049	0.233	0.045	0.252

TABLE II. SIGNAL AMPLITUDES AFTER SEVERAL GA ITERATIONS (UNTIL IT REACHES STATIONARY POINT) AND AFTER THE PROPOSED RECONSTRUCTION

Signal amplitudes from example 1			Signal amplitudes from example 2		
After 13 iter. GA	After recon. with the proposed method	Exact	After 16 iter. of GA	After recon. with the proposed method	Exact
3.3	3.5	3.5	1.00	1.20	1.20
2.8	3	3	0.98	1.10	1.10
1.8	2	2	1.13	1.30	1.30
0.8	1	1	1.34	1.50	1.50
3.9	4	4	1.24	1.40	1.40
x	x	x	0.32	0.47	0.47
x	x	x	0.79	0.90	0.90
x	x	x	0.35	0.39	0.39

V. CONCLUSION

The paper proposes a method for signal reconstruction in the CS scenario. The method combines the gradient algorithm, the threshold calculation for detection components positions and the optimization problem solving for recovering signals' amplitudes. Compared with the gradient algorithm, the proposed method increases reconstruction speed, which is proved experimentally. The accuracy of the proposed method is verified by measuring the amplitude values of the reconstructed components.

REFERENCES

- [1] Y. C. Eldar G. Kutyniok, "Compressed Sensing: Theory and Applications", Cambridge University Press, May 2012.
- [2] Y. C. Eldar "Sampling Theory: Beyond Bandlimited Systems", Cambridge University Press, April 2015.
- [3] S. Stanković, I. Orović, E. Sejdović, "Multimedia Signals and Systems: Basic and Advance Algorithms for Signal Processing," Springer, 2016.
- [4] D. Donoho, "Compressed sensing," IEEE Trans. on Information Theory, 52, (4), pp. 1289 – 1306, 2006.
- [5] R. Baraniuk, "Compressive sensing," IEEE Signal Processing Magazine, 24, (4), pp. 118-121, 2007.

- [6] S. Stanković, I. Orović, Lj. Stanković, "An Automated Signal Reconstruction Method based on Analysis of Compressive Sensed Signals in Noisy Environment," Sig. Proc., vol. 104, Nov 2014.
- [7] E. J. Candes, and M. Wakin, "An Introduction to Compressive Sampling", IEEE Sig. Proc. Mag., vol. 25, no. 2, pp. 21-30, March 2008.
- [8] C. Bernard, C. Ioana, I. Orovic, S. Stankovic, "Analysis of underwater signals with nonlinear time-frequency structures using warping based compressive sensing algorithm," MTS/IEEE North American OCEANS conference, October 2015, Washington, DC, United States.
- [9] M. Elad, Sparse and Redundant Representations: From Theory to Applications in Signal and Image Processing, Springer, 2010.
- [10] LJ. Stankovic, S. Stankovic, M. Amin, "Missing Samples Analysis in Signals for Applications to L-estimation and Compressive Sensing," Signal Processing, vol. 94, Jan 2014, pp. 401-408, 2014.
- [11] G. Pope, "Compressive Sensing: a Summary of Reconstruction Algorithms", Eidgenossische Technische Hochschule, Zurich, 2008.
- [12] S. Bahmani, "Algorithms for Sparsity-Constrained Optimization", Springer Theses, Series Volume 261, ISBN 978-3-319-01880-5, 2014
- [13] A. Draganic, I. Orovic, S. Stankovic, "On some common compressive sensing recovery algorithms and applications - Review paper," Facta Universitatis, Series: Electronics and Energetics, Vol 30, No 4, 2017.
- [14] I. Stankovic, I. Orovic, S. Stankovic, M. Dakovic, "Iterative Denoising of Sparse Images," 39th International Convention on Information and Communication Technology, Electronics and Microelectronics, (MIPRO 2016), 2016.
- [15] T. Blumensath, M. E. Davies, "Iterative Thresholding for Sparse Approximations", Journal of Fourier Analysis and Applications, vol. 14, no. 5-6, pp 629-654, December 2008.
- [16] M. Brajovic, I. Orovic, M. Dakovic, S. Stankovic, "Gradient-based signal reconstruction algorithm in the Hermite transform domain," Electronic Letters, Volume: 52, Issue: 1, pp. 41 - 43, 2015.
- [17] T. Zhang, "Sparse Recovery with Orthogonal Matching Pursuit Under RIP," IEEE Trans. on Information Theory, 57(9), 2011, pp. 6215-6221
- [18] I. Orovic, A. Draganic, N. Lekic, S. Stankovic, "A System for Compressive Sensing Signal Reconstruction," 17th IEEE International Conference on Smart Technologies EUROCON 2017, Ohrid, Macedonia
- [19] S. Stankovic, I. Orovic, LJ. Stankovic, A. Draganic, "Single-Iteration Algorithm for Compressive Sensing Reconstruction," Telfor Journal Vol. 6, No. 1, pp. 36-41, 2014.
- [20] S. Vujovic, M. Dakovic, I. Orovic, S. Stankovic, "An Architecture for Hardware Realization of Compressive Sensing Gradient Algorithm," 4th Mediterranean Conference on Embedded Computing MECO - 2015, Budva, Montenegro, 2015.
- [21] R. G. Baraniuk, T. Goldstein, A. C. Sankaranarayanan, C. Studer, A. Veeraraghavan, M. B. Wakin, "Compressive Video Sensing: Algorithms, architectures, and applications," in IEEE Signal Processing Magazine, vol. 34, no. 1, pp. 52-66, Jan. 2017
- [22] S. Stankovic, S. Vujovic, I. Orovic, M. Dakovic, LJ. Stankovic, "Combination of Gradient Based and Single Iteration Reconstruction Algorithms for Sparse Signals," 17th IEEE International Conference on Smart Technologies, EUROCON 2017, Ohrid, Macedonia.
- [23] J. A. Tropp, A. C. Gilbert, "Signal Recovery From Random Measurements Via Orthogonal Matching Pursuit," IEEE Transaction on Information Theory, vol. 53, no.12, 2007.
- [24] X. Li, G. Bi, "Time-frequency representation reconstruction based on the compressive sensing", 2014 9th IEEE Conference on Industrial Electronics and Applications, Hangzhou, 2014, pp. 1158-1162
- [25] S. Stankovic, I. Orovic, "An Approach to 2D Signals Recovering in Compressive Sensing Context," Circuits Systems and Signal Processing, April 2017, Volume 36, Issue 4, pp. 1700-1713.
- [26] LJ. Stanković, M. Daković, "On a Gradient-Based Algorithm for Sparse Signal Reconstruction in the Signal/Measurements Domain" Mathematical Problems in Engineering, vol. 2016, Article ID 6212674.